Girard Paradox

Nguyen Ngoc Khanh

May 27, 2026

This is a short note summarizing the proof of Martin-Löf [1] on Girard Paradox. Formally, a system admitting the type of all types is inconsistent, that motivates the use of $\mathrm{Type}\, n : \mathrm{Type}(n+1)$ in Martin-Löf Type Theory

We will use the language of set in the summary for familiarity.

Firstly, we introduce the notion of poset used in this document

Definition 1 (poset)

A poset X is a set equipped with a transitive relation $<$, that is

$$x < y \text{ and } y < z \text{ implies } x < z$$

for every $x, y, z \in X$

Definition 2 (well-founded)

A poset X is said to be well-founded if there is no infinitely decending chain

$$... < x_n < x_{n-1} < ... < x_1 < x_0$$

We write $\mathrm{wf}\, X$

Note that, if X is a set and $x < x$ for some $x \in X$, then X cannot be well-founded poset. Since $x < x$ introduces an infinite decending chain

$$... < x < x$$

Lemma 3

if X is a set and $x < x$ for some $x \in X$, then X cannot be well-founded poset

Let $\mathcal{P}$ be the collection of well-founded posets.

$$\mathcal{P} = \{S : \mathrm{Set} \mid \mathrm{wf}\, S\}$$

Assuming $\mathcal{P}$ is a set $\mathcal{P} : \mathrm{Set}$, we can construct a transitive relation $\prec$ on $\mathcal{P}$ as follows: for every $S, R \in \mathcal{P}$, $S \prec R$ if and only if there exists an order preserving map $f : S \to R$ such that there exists an element $r \in R$ that dominates all elements in the image of f, that is for every $x \in \mathrm{im}\, f \subseteq R$, $x < r$.

Now, assuming the partially ordered set $\mathcal{P}$ (with $\prec$) is not well-founded, that is, there exists an infinite chain in $\mathcal{P}$

$$... \prec S_n \prec S_{n-1} \prec ... \prec S_1 \prec S_0$$

Let denote the map $f_n : S_{n+1} \to S_n$ and $s_n \in S_n$ corresponding to the relation $S_{n+1} \prec S_n$. Then, for every n, for every $x \in \operatorname{im} f_n$, $x < s_n$, that implies

$$s_{n-1} < s_n \text{ in } S_n$$

We can construct an infinitely decending chain in S_0 by

$$... < s_n < s_{n-1} < ... < s_1 < s_0$$

This contradicts the assumption that all elements of $\mathcal{P}$ being well-founded. Therefore, if $\mathcal{P}$ is a set, it must be well-founded, hence $\mathcal{P} \in \mathcal{P}$. Next, we will write $P := \mathcal{P}$ in the context where $\mathcal{P}$ is an element of $\mathcal{P}$.

Now, we will show that $S \prec P$ for every $S \in \mathcal{P}$

Definition 4 (initial segment)
Let S be a poset, the initial segment of S at $r \in S$ is defined by

$$S_r = \{y \in S : y < r\}$$

If S is well-founded, then S_r is also a well-founded poset.

For every $S \in \mathcal{P}$, define the map

$$\begin{aligned} g_S : S &\to \mathcal{P} \\ r &\mapsto S_r \end{aligned}$$

This map makes $S \prec P$. In particular, $P \prec P$ in $\mathcal{P}$, lemma 3 leads to contradiction.

A formal proof of this statement can be found on my github or lean4web

References

[1] P Martin-Löf. An intuitionistic theory of. In *Types: Predicative Part, Logic Colloquium*, volume 73, 1972.