sheaf isomorphism

ismorphisms of sheaves on open cover implies isomorphism on the whole sheaves

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Lemma 1. *Let $E \to X$ and $F \to X$ be vector bundles over X. Then there is a sheaf isomorphism*

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(F) \to \mathcal{E}(E \otimes F)$$

where $\mathcal{E}(E)$ and $\mathcal{E}(F)$ are sheaves induced from sections of vector bundles.

Proof. For every $x \in X$, pick $U \subseteq X$ containing x small enough such that $E|_U \to U$ and $F|_U \to U$ are trivial bundles. Let

$$\mathcal{T}(-) = \mathcal{E}(E)(-) \otimes_{\mathcal{E}(-)} \mathcal{E}(F)(-)$$

be the tensor product of presheaves. We will construct the map t and show that the diagram below commutes

$$\begin{tikzcd} \mathcal{T}(U) \arrow[r, "t"] \arrow[d, "r_V^U"] & \mathcal{E}(E \otimes F)(U) \arrow[d, "r_V^U"] \\ \mathcal{T}(V) \arrow[r, "t"] & \mathcal{E}(E \otimes F)(V) \end{tikzcd}$$

Let $e = \{e_1, e_2, ..., e_m\}$ and $f = \{f_1, f_2, ..., f_n\}$ be frames of E and F on U. Then, every element $\xi \in \mathcal{T}(U)$ can be written as

$$\xi = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} (e_i \otimes_{\mathcal{E}(U)} f_j)$$

where $\xi_{ij} \in \mathcal{E}(U)$. And every element $\eta \in \mathcal{E}(E \otimes F)(U)$ can be written as

$$\eta(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \eta_{ij}(x)(e_i(x) \otimes f_i(x))$$

where $\eta_{ij} \in \mathcal{E}(U)$. Hence, there exists a natural isomorphism of sheaves from $\mathcal{T}|_U = \mathcal{T}^{sh}|_U$ to $\mathcal{E}(E \otimes F)|_U$ defined on $U \subseteq X$.

Lemma 2. *If $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X and there exists an open cover $\{U_i\}_{i \in I}$ for X such that $\tau_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ is an isomorphism of sheaves for all $i \in I$, then there exists an isomorphism of sheaves $\tau : \mathcal{F} \to \mathcal{G}$*

Proof. Let V be open, without loss of generality, we can assume that there exists an open cover $\{U_i\}_{i \in I}$ for V such that for each $U_i \in \{U_i\}_{i \in I}$, $\tau_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ is an isomorphism of sheaves. Let $U \in \{U_i\}$,

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "r_U^V"] \arrow[d, dashed, "\tau_V"'] & \mathcal{F}(U) \arrow[d, "\tau_U"] \\ \mathcal{G}(V) \arrow[r, "r_U^V"] & \mathcal{G}(U) \end{tikzcd}$$

Let $f \in \mathcal{F}(U)$, define

$$g_U = \tau_U r_U^V f$$

For any $U_i, U_j \in \{U_i\}_{i \in I}$ with $U_i \cap U_j \neq \varnothing$, the diagram below commutes

$$
\begin{tikzcd}
\mathcal{F}(V) \arrow[rr, "r^V_{U_i}"] \arrow[dr, "r^V_{U_j}"] \arrow[dd, dashed, "\tau_V"'] & & \mathcal{F}(U_i) \arrow[dr, "r^{U_i}_{U_i \cap U_j}"] \arrow[dd, "\tau_{U_i}"] & \\
& \mathcal{F}(U_j) \arrow[rr, "r^{U_j}_{U_i \cap U_j}"] \arrow[dd, "\tau_{U_j}"] & & \mathcal{F}(U_i \cap U_j) \arrow[dd, "\tau_{U_i \cap U_j}"] \\
\mathcal{G}(V) \arrow[rr] \arrow[dr] & & \mathcal{G}(U_i) \arrow[dr, "r^{U_i}_{U_i \cap U_j}"] & \\
& \mathcal{G}(U_j) \arrow[rr, "r^{U_j}_{U_i \cap U_j}"] & & \mathcal{G}(U_i \cap U_j)
\end{tikzcd}
$$

Hence, by commutativity, we have

$$r^{U_i}_{U_i \cap U_j} g_{U_i} = r^{U_j}_{U_i \cap U_j} g_{U_j}$$

By definition of sheaf, there exists a unique map $\tau_V : \mathcal{F}(V) \to \mathcal{G}(V)$ that makes the diagram commutes. Similarly, we can construct a unique map $\tau_V^{-1} : \mathcal{G}(V) \to \mathcal{F}(V)$. It can be verified that τ defines a natural isomorphism between two functors $\mathcal{F}$ and $\mathcal{G}$ ☐

Back to the proof, use the lemma, we can construct a natural isomorphism

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(F) \to \mathcal{E}(E \otimes F)$$

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