

# complex

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*this is the note for my learning on complex analysis*

## 1 Definition of complex numbers

**Definition 1** (Real Numbers). *An ordered field  $R$  satisfies least upper bound property: Every non-empty subset of  $R$  with an upper bound has a least upper bound.*

**Definition 2** (Complex Numbers). *The set of complex numbers is defined as  $\mathbb{C} = \mathbb{R} \times \mathbb{R}$  with additional structure*

- *Complex Addition (+):  $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$*
- *Complex Multiplication ( $\cdot$ ):  $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$*

*$(\mathbb{C}, +, \cdot)$  form the field of complex numbers.*

**Theorem 1** (Real sub-field of Complex Numbers). *The set  $R = \{(a, 0) : a \in \mathbb{R}\} \subset \mathbb{C}$  together with Complex Addition and Complex Multiplication is Real Numbers*

In fact, Given any non-zero complex number  $(a, b)$  where  $a, b \neq 0$ , any set of the form  $\{(\alpha a, \alpha b) : \alpha \in \mathbb{R}\}$  is also Real Numbers.

In the context of Complex Numbers, we only call the set  $R = \{(a, 0) : a \in \mathbb{R}\}$  Real Numbers and write them as  $a = (a, 0)$

### 1.1 Representation of Complex Numbers

#### 1.1.1 Standard form

**Theorem 2** (Standard form of Complex Numbers). *Any complex number can be written as  $a + ib$  where  $a, b \in R$  and  $i = (0, 1)$ .*

Proof

$$\begin{aligned}
a + ib &= (a, 0) + (b, 0) \cdot (0, 1) && \text{(rewrite)} \\
&= (a, 0) + (b0 - 01, b1 + 00) && \text{(complex multiplication)} \\
&= (a, 0) + (0, b) && \text{(the field of real numbers)} \\
&= (a, b) && \text{(complex addition)}
\end{aligned}$$

From now on, instead of writing a complex number as  $(a, b)$ , we will write them in standard form:  $a + ib$

### 1.1.2 Polar form

Define the equivalent relation  $\sim$  in the set of real numbers with element denoted as  $\theta \in \mathbb{R}$

**Definition 3** ( $2\pi$  periodic).

$$\theta \sim \theta + 2\pi$$

Let the equivalent class  $\Theta$  be defined by the equivalent relation  $\sim$  on  $\mathbb{R}$ . Element of  $\Theta$  is denoted as  $[\theta]$

**Definition 4** (Polar form of Complex Numbers). *Any non-zero complex number can be written in polar form*

$$(r, [\theta]) \in (0, \infty) \times \Theta$$

where  $r = \sqrt{a^2 + b^2}$  and  $\theta = \arctan(b/a)$

Given  $(r, [\theta])$  in polar form, we can rewrite it in standard form  $a + ib$  where  $a = r \cos \theta$ ,  $b = r \sin \theta$

#### Multiplication in polar form

By the special structure of polar form, multiplication is simpler

$$(r_1, [\theta_1]) \cdot (r_2, [\theta_2]) = (r_1 r_2, [\theta_1 + \theta_2])$$

## 1.2 Complex function

*several functions defined for the set of complex numbers*

### 1.2.1 Modulus - Argument - Conjugate

**Modulus**  $|\cdot|$

$$|a + ib| = \sqrt{a^2 + b^2}$$

Complex Numbers inherits the L2 norm in  $\mathbb{R}^2$  making it a normed space.

**Argument**  $\arg$

$$\arg(a + ib) = \arctan(b/a)$$

where  $\arctan(b/a)$  is defined to be in the range  $[0, 2\pi)$  or  $(-\pi, \pi]$

**Conjugate** :

$$\overline{a + ib} = a - ib$$

### 1.2.2 Inverses

**Additive inverse**

$$-(a + ib) = -a - ib$$

**Multiplicative inverse**

$$(a + ib)^{-1} = \frac{a - ib}{a^2 + b^2}$$

### 1.2.3 Exponential

**Definition 5** (Exponential function).

$$e^{a+ib} = e^a(\cos b + i \sin b)$$

**Theorem 3** (Polar form as Exponential).

$$(r, [\theta]) = re^{i\theta}$$

Proof

$$\begin{aligned} re^{i\theta} &= r(\cos \theta + i \sin \theta) && \text{(Exponential)} \\ &= r \cos \theta + ir \sin \theta && \text{(the field of complex numbers)} \\ &= (r, [\theta]) && \text{(polar form definition)} \end{aligned}$$

From now on, instead of writing a complex number in polar form as  $(r, [\theta])$ , we will write them as complex exponential:  $re^{i\theta}$

**Inverse of Exponential**

Exponential function is  $2\pi$  periodic in imaginary axis, i.e

$$e^{a+ib} = e^{a+i(b+2\pi)}$$

Hence the inverse of exponential is defined as a multi-function

**Definition 6** (Logarithm). Defining  $\log$  as a multi-function being the inverse of exponential function

$$\begin{aligned}\log : \mathbb{C} \setminus \{0\} &\rightarrow \mathcal{P}(\mathbb{C}) \\ \log(z) &= \{x : x \in \mathbb{C} \wedge e^x = z\} \\ \log(re^{i\theta}) &= \{\log r + i(\theta + 2\pi k) : k \in \mathbb{Z}\}\end{aligned}$$

**Definition 7** (Power function).<sup>1</sup>

$$z^n = e^{n \log z}$$

Write  $z$  in polar form

$$\begin{aligned}z^n &= (re^{i\theta})^n \\ &= e^{n \log(re^{i\theta})} \\ &= e^{n\{\log r + i(\theta + 2\pi k) : k \in \mathbb{Z}\}} \\ &= e^{n \log r} e^{in(\theta + 2\pi k) : k \in \mathbb{Z}}\end{aligned}$$

**$n$  is a positive integer**

$z^n : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  is a proper function<sup>2</sup>

$$\begin{aligned}z^n &= e^{n \log r} e^{in(\theta + 2\pi k) : k \in \mathbb{Z}} \\ &= r^n e^{in\theta}\end{aligned}$$

**$m$ -th root, i.e  $n = 1/m$  where  $m \in \mathbb{N} \setminus \{0\}$**

There are  $m$   $m$ -th roots of complex number  $z \neq 0$

$$\begin{aligned}z^n &= \sqrt[m]{z} = e^{\frac{\log r}{m}} e^{i(\frac{\theta}{m} + \frac{2\pi}{m}k) : k \in \mathbb{Z}} \\ &= \sqrt[m]{r} \{e^{i(\frac{\theta}{m} + \frac{2\pi}{m}k)} : k \in \mathbb{Z} \cup [0, m)\}\end{aligned}$$

**$n$  is irrational**

There are infinitely many output values of power function

**Properties of Exponential function and Power function**

Exponential function

<sup>1</sup>in this note, given  $f : A \rightarrow B$ , we write  $f(\overline{A}) = \{f(x) : x \in \overline{A}\}$  where  $\overline{A} \subseteq A$

<sup>2</sup>one input, one output

- $e^{a+b} = e^a e^b$
- $e^{-a} = (e^a)^{-1}$

Power function

- $z^{a+b} = z^a z^b$
- $z^{-a} = (z^a)^{-1}$
- $z^{ab} = (z^a)^b$

## 2 Subspace of $\mathbb{R}^{2 \times 2}$

The field of complex numbers is isomorphic to a subspace of the vector space of  $(2 \times 2)$  matrices with real entries, i.e.  $\mathbb{R}^{2 \times 2}$  with additional structure as follows  
Basis of the complex subspace is  $\mathbf{B} = \{r, i\}$  where

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

All complex numbers is represented as  $z = ar + ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  where  $a, b \in \mathbb{R}$ .

Furthermore, we define addition as matrix addition and multiplication as matrix multiplication.

### 2.1 Polar form

Let  $r = \sqrt{a^2 + b^2}$  and  $\theta = \arctan(b/a)$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

*Geometrically speaking, a complex number is a composition of a scaling operator and a rotation operator in  $\mathbb{R}^2$*