# MA4271 Homework 1

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**Problem 1** *Let $\alpha : I \to \mathbb{R}^3$ be a parameterized smooth curve with $||\alpha(t)|| \neq 0$ and $||\alpha'(t)|| \neq 0$ for all $t \in I$. Prove that $||\alpha(t)||^2$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$*

$(\implies)$
Let $||\alpha(t)||^2 = c > 0$, we have

$$c = ||\alpha(t)||^2 = \alpha(t) \cdot \alpha(t)$$

Take the derivative on both sides.

$$\begin{aligned} 0 &= \frac{d}{dt}(\alpha(t) \cdot \alpha(t)) \\ &= \alpha(t) \cdot \alpha'(t) + \alpha'(t) \cdot \alpha(t) && \text{(property of dot product derivative)} \\ &= 2(\alpha'(t) \cdot \alpha(t)) && \text{(symmetry of dot product)} \end{aligned}$$

The inner product between $\alpha'(t)$ and $\alpha(t)$ is zero; that is, they are orthogonal for all $t \in I$
$(\impliedby)$

$$0 = \alpha'(t) \cdot \alpha(t) = \frac{1}{2}\frac{d}{dt}||\alpha(t)||^2$$

$||\alpha(t)||^2$ is a real-valued function having derivative being 0 for all $t \in I$, therefore, it is a constant

**Problem 2** *Prove that the equation of a plane passing through three noncolinear*[1] *points $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2), p_3 = (x_3, y_3, z_3)$ is given by*

$$(p - p_1) \times (p - p_2) \cdot (p - p_3) = 0$$

*where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$ is the vector $(x - x_1, y - y_1, z - z_1)$*

[1] This means that $p_1, p_2, p_3$ cannot be covered by a single line

For any 3 vectors in $\mathbb{R}^3$, $(u \times v) \cdot w = \det(u, v, w)$

$(\implies)$

$[(p - p_1) \times (p - p_2)] \cdot (p - p_3) = 0$ implies $\det(p - p_1, p - p_2, p - p_3) = 0$. So, 3 vectors $p - p_1, p - p_2, p - p_3$ lie on a subspace of dimension $d < 3$. Since $p_1, p_2, p_3$ is noncolinear, $d > 1$. Therefore, $d = 2$; that is, the set of all $p$ lie on the plane containing $p_1, p_2, p_3$

$(\impliedby)$

Given $p$ on the plane containing $p_1, p_2, p_3$, the span of $p - p_1, p - p_2, p - p_3$ must be a proper subspace of $\mathbb{R}^3$; that is, $\det(p - p_1, p - p_2, p - p_3) = 0$. Hence, $[(p - p_1) \times (p - p_2)] \cdot (p - p_3) = 0$

**Problem 3** *Let $u(s) = (u_1(s), u_2(s), u_3(s)), v(s) = (v_1(s), v_2(s), v_3(s))$ be smooth maps from interval $(a, b)$ into $\mathbb{R}^3$. If the derivative $u'(s)$ and $v'(s)$ satisfy the conditions*

$$u'(s) = c_1 u(s) + c_2 v(s), \quad v'(s) = c_3 u(s) - c_1 v(s)$$

*where $c_1, c_2, c_3 \in \mathbb{R}$ are constants. Prove that $u(s) \times v(s)$ is a constant vector.*

$$\begin{aligned}
\frac{d}{ds}(u \times v) &= \frac{du}{ds} \times v + u \times \frac{dv}{dt} && \text{(property of vector product derivative)} \\
&= (c_1 u + c_2 v) \times v + u \times (c_3 u - c_1 v) && \text{(premise)} \\
&= c_1 u \times v + c_2 v \times v + c_3 u \times u - c_1 u \times v && \text{(linearity of vector product)} \\
&= c_1 u \times v - c_1 u \times v && (u \times u = 0, v \times v = 0) \\
&= 0
\end{aligned}$$

All 3 components of $u \times v$ is a real-valued function having derivative being 0 for all $s \in (a, b)$, therefore $u \times v$ is a constant vector.

**Problem 4** *Let $\alpha : I \to \mathbb{R}^3$ be a smooth regular curve parameterized by arc length and $\alpha''(s) \neq 0$ for all $s \in I$. Prove that the torsion $\tau$ of $\alpha$ is given by*

$$\tau(s) = -\frac{\alpha'(s) \times \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}$$

$\alpha$ is parameterized by arc length, then

- tangent vector $\alpha'$ has norm $||\alpha'|| = 1$
- normal vector $n = \frac{\alpha''}{||\alpha''||}$, curvature $\kappa = ||\alpha''||$
- binormal vector $b = \alpha' \times n$

We have

$$
\begin{aligned}
\tau n &= b' \\
&= \frac{d}{ds}(\alpha' \times n) \\
&= \alpha'' \times \frac{\alpha''}{\kappa} + \alpha' \times \frac{dn}{ds} && \text{(property of vector product derivative)} \\
&= \alpha' \times \frac{dn}{ds} && (\alpha'' \times \alpha'' = 0) \\
&= \alpha' \times \frac{d}{ds}\frac{\alpha''}{\kappa} \\
&= \alpha' \times \left(\frac{\alpha'''}{\kappa} + \gamma\alpha''\right) && \text{where } \gamma = \frac{d}{ds}\frac{1}{\kappa} \\
&= \frac{\alpha' \times \alpha'''}{\kappa} + \gamma(\alpha' \times \alpha'') \\
&= \frac{\alpha' \times \alpha'''}{\kappa} + \gamma\kappa(\alpha' \times n) \\
&= \frac{\alpha' \times \alpha'''}{\kappa} + \gamma\kappa b
\end{aligned}
$$

Dot product both sides by $n$

$$\tau = \tau(n \cdot n) = \frac{(\alpha' \times \alpha''') \cdot n}{\kappa} + \gamma\kappa(b \cdot n)$$

$b \cdot n = 0$ implies

$$
\begin{aligned}
\tau &= \frac{(\alpha' \times \alpha''') \cdot n}{\kappa} \\
&= \frac{(\alpha' \times \alpha''') \cdot \alpha''}{\kappa^2} \\
&= \frac{\det(\alpha', \alpha''', \alpha'')}{\kappa^2} \\
&= -\frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2} && \text{(odd permutation} \rightarrow \text{even permutation)} \\
&= -\frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\kappa^2}
\end{aligned}
$$

**Problem 5** *Let $\alpha : I \to \mathbb{R}^3$ be a smooth regular curve parameterized by arc length. Assume that $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all $s \in I$. Prove that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that*

$$(R(s))^2 + (R'(s))^2(T(s))^2 = const$$

*where $R(s) = \frac{1}{\kappa(s)}, T(s) = \frac{1}{\tau(s)}$ and $R'(s)$ is the derivative of $R(s)$ relative to $s$*

In this proof, we will make an additional assumption $\kappa(s) \neq 0$ for all $s$. We have

$$(R(s))^2 + (R'(s))^2(T(s))^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2$$

**Lemma 1** *A trajectory of a smooth regular curve $\alpha$ parameterized by arc length lies on a sphere centered at $c$ of radius $r > 0$ with $\tau(s) \neq 0$ and $\kappa(s) \neq 0, \kappa'(s) \neq 0$ for all $s$. Must have*

$$\alpha' \cdot (\alpha - c) = 0 \tag{1}$$

$$n \cdot (\alpha - c) = -\frac{1}{\kappa} \tag{2}$$

$$b \cdot (\alpha - c) = -\frac{\kappa'}{\tau\kappa^2} \tag{3}$$

*where $\kappa$ is curvature, $\tau$ is torsion, $n$ is normal vector, and $b$ is binormal vector.*

**Main Proof**

$(\implies)$

Let $c$ be the center of the sphere with radius $r > 0$ where the trajectory of $\alpha$ lies on. Project $\alpha - c$ onto *Frenet trihedron*

$$\alpha - c = (\alpha' \cdot (\alpha - c))\alpha' + (n \cdot (\alpha - c))n + (b \cdot (\alpha - c))b$$
$$\alpha - c = -\frac{1}{\kappa}n - \frac{\kappa'}{\tau\kappa^2}b$$

Take the norm of both sides

$$const = ||a - c|| = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2$$

$(\impliedby)$

Let $\beta$ be defined by

$$\beta = \alpha + \frac{1}{\kappa}n + \frac{\kappa'}{\tau\kappa^2}b$$

We will prove that $\beta$ is a constant w.r.t arc length $s$. Hence, the norm of $\alpha - \beta$ is

$$||a - \beta|| = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2 = const$$

We will conclude that $\beta$ is the center of the sphere where the trajectory of $\alpha$ lies on.

Take the derivative w.r.t $s$ of $\beta$

$$
\begin{aligned}
\beta' &= \alpha' + \left[-\frac{\kappa'}{\kappa^2}n + \frac{1}{\kappa}(-\kappa\alpha' - \tau b)\right] + \left[\frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right)b + \frac{\kappa'}{\tau\kappa^2}(\tau n)\right] \\
&= \alpha' + \left[\frac{1}{\kappa}(-\kappa\alpha' - \tau b)\right] + \left[\frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right)b\right] && \text{(remove } -\frac{\kappa'}{\kappa^2}n) \\
&= \left[\frac{1}{\kappa}(-\tau b)\right] + \left[\frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right)b\right] && \text{(remove } \alpha') \\
&= \left[-\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right)\right]b
\end{aligned}
$$

On the other hand,

$$const = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2$$

Take the derivative w.r.t to $s$ on both sides

$$
\begin{aligned}
0 &= \left[2\frac{1}{\kappa}\frac{-\kappa'}{\kappa^2}\right] + \left[2\frac{\kappa'}{\tau\kappa^2}\frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right)\right] \\
&= -\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{\kappa'}{\tau\kappa^2}\right) && \text{(divide both sides by } 2\frac{\kappa'}{\tau\kappa^2})
\end{aligned}
$$

That implies $\beta' = 0$. As discussed earlier, $||a - \beta|| = const$. Therefore, the trajectory of $\alpha$ lies a sphere centered at $\beta$

**Proof of Lemma ??** We have

$$||a - c||^2 = r^2$$

Take the derivative w.r.t $s$ on both sides

$$
\begin{aligned}
2(\alpha - c)' \cdot (\alpha - c) &= 0 && \text{(derivative of dot product formula)} \\
\alpha' \cdot (\alpha - c) &= 0
\end{aligned}
$$

We have the first equation

$$\alpha' \cdot (\alpha - c) = 0$$

Take the derivative w.r.t $s$ on both sides

$$
\begin{aligned}
\alpha'' \cdot (\alpha - c) + \alpha' \cdot (\alpha - c)' &= 0 \\
\kappa n \cdot (\alpha - c) + \alpha' \cdot \alpha' &= 0 && \text{(definition of curvature)} \\
n \cdot (\alpha - c) &= -\frac{1}{\kappa}
\end{aligned}
$$

We have the second equation

$$n \cdot (\alpha - c) = -\frac{1}{\kappa}$$

Take the derivative w.r.t $s$ on both sides

$$
\begin{aligned}
n' \cdot (\alpha - c) + n \cdot (\alpha - c)' &= -(-1)\frac{\kappa'}{\kappa^2} \\
(-\kappa\alpha' - \tau b) \cdot (\alpha - c) + n \cdot \alpha' &= \frac{\kappa'}{\kappa^2} && (\textit{Frenet formulas}) \\
(-\kappa\alpha' - \tau b) \cdot (\alpha - c) + 0 &= \frac{\kappa'}{\kappa^2} && (n \text{ and } \alpha' \text{ are orthogonal}) \\
\kappa\alpha' \cdot (\alpha - c) + \tau b \cdot (\alpha - c) &= -\frac{\kappa'}{\kappa^2} && \text{(linearity of dot product)} \\
\tau b \cdot (\alpha - c) &= -\frac{\kappa'}{\kappa^2} && (\alpha' \cdot (\alpha - c) = 0) \\
b \cdot (\alpha - c) &= -\frac{\kappa'}{\tau\kappa^2}
\end{aligned}
$$

We have the third equation

$$b \cdot (\alpha - c) = -\frac{\kappa'}{\tau\kappa^2}$$