# MA4271 Homework 3

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## 1 Problem

**Problem 1** *Prove that the absolute value of the torsion $\tau$ at a point of an asymptotic curve whose curvature is nowhere zero is given by*

$$|\tau| = \sqrt{-K}$$

*where $K$ is the Gaussian curvature of the surface at the given point.*

Let $\alpha(s)$ be an asymptotic curve on surface $S$ whose curvature is nowhere zero. The normal curvature of $\alpha(s)$ is

$$\langle N, \kappa n \rangle = 0$$

where $N$ denotes the unit normal vector of surface, $n$ denotes the unit normal vector of $\alpha(s)$, $\kappa$ denotes the curvature of $\alpha$. As $\kappa \neq 0$, $n$ exists everywhere. Torsion of $\alpha$ is defined as $b'(s) = \tau(s)n(s)$. Restrict $N$ on $\alpha$, as $N(s)$ is orthogonal to $n(s)$ and $\alpha'(s) \in T_{\alpha(s)}S$, we have

$$N(s) = \lambda(n(s) \times \alpha'(s)) = \lambda b(s)$$

where $\lambda$ is either $+1$ or $-1$. Then

$$\tau(s) = ||b'(s)|| = ||N'(s)||$$

We can write $N'(s)$ in term of Gauss map (differential of $N$ along the direction of tangent vector)

$$\begin{aligned} N'(s) &= dN_{\alpha(s)}(\alpha'(s)) \\ &= dN_{\alpha(s)}(e_1 \cos\theta + e_2 \sin\theta) && (\text{let } \alpha'(s) = e_1 \cos\theta + e_2 \sin\theta) \\ &= e_1 k_1 \cos\theta + e_2 k_2 \sin\theta && (\text{eigen direction}) \end{aligned}$$

Then $||N'(s)||^2 = k_1^2(\cos\theta)^2 + k_2^2(\sin\theta)^2$. On the other hand, by Euler formula, $0 = k_1(\cos\theta)^2 + k_2(\sin\theta)^2$, Then

$$\begin{aligned} ||N'(s)||^2 &= k_1k_1(\cos\theta)^2 + k_2k_2(\sin\theta)^2 \\ &= k_1(-k_2(\sin\theta)^2) + k_2(-k_1(\cos\theta)^2) \\ &= -k_1k_2 = -K \end{aligned}$$

Then

$$\tau(s) = ||N'(s)|| = \sqrt{-K}$$

**Problem 2** *Suppose that $S_1$ and $S_2$ intersect along a regular curve $C$ and make an angle $\theta(p), p \in C$. Assume that $C$ is a line of curvature of $S_1$. Prove that $\theta(p)$ is constant if and only if $C$ is a line of curvature of $S_2$*

$(\Longleftarrow)$ $C = \alpha(s)$ is the line of curvature of both $S_1$ and $S_2$, by Oline Rodrigues,

$$\begin{aligned} N_1'(s) &= \lambda_1(s)\alpha'(s) \\ N_2'(s) &= \lambda_2(s)\alpha'(s) \end{aligned}$$

where $N_1(s), N_2(s)$ is the restriction of unit normal vectors of $S_1, S_2$ on $C$, $-\lambda_1(s), -\lambda_2(s)$ are the corresponding principle curvatures

As $\alpha'(s)$ is orthogonal to $N_1(s)$ and $N_2(s)$

$$\langle N_1'(s), N_2(s)\rangle = \lambda_1(s)\langle \alpha'(s), N_2(s)\rangle = 0$$
$$\langle N_2'(s), N_1(s)\rangle = \lambda_2(s)\langle \alpha'(s), N_1(s)\rangle = 0$$

Then

$$\frac{d}{ds}\langle N_1(s), N_2(s)\rangle = \langle N_1'(s), N_2(s)\rangle + \langle N_2'(s), N_1(s)\rangle = 0$$

Then $\theta = \langle N_1, N_2\rangle$ is constant along $C$

$(\Longrightarrow)$ $C = \alpha(s)$ is the line of curvature of $S_1$, and $\theta = \langle N_1, N_2\rangle$ is constant along $C$, similarly by Oline Rodrigues

$$N_1'(s) = \lambda_1(s)\alpha'(s)$$

Then

$$\langle N_1'(s), N_2(s)\rangle = \lambda_1(s)\langle \alpha'(s), N_2(s)\rangle = 0$$

Then

$$\langle N_2'(s), N_1(s)\rangle = \frac{d}{ds}\langle N_1(s), N_2(s)\rangle - \langle N_1'(s), N_2(s)\rangle = 0$$

As $N_2(s)$ is a unit vector, then $N_2'(s)$ is orthogonal to $N_2(s)$. Since both $\alpha'(s)$ and $N_2'(s)$ are orthogonal two both $N_1(s)$ and $N_2(s)$, then they are parallel, i.e. $N_2'(s) = \lambda(s)\alpha'(s)$. By Oline Rodrigues, $C$ is the line of curvature of $S_2$

**Problem 3** *Let $\lambda_1, ..., \lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, 2\pi/m, ..., (m-1)2\pi/m$ with a principle direction. Prove that*

$$\lambda_1 + ... + \lambda_m = mH$$

*where $H$ is the mean curvature at $p$*

Denote $\theta_i = 2\pi\frac{i-1}{m}$, by Euler formula,

$$\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} [k_1(\cos\theta_i)^2 + k_2(\sin\theta_i)^2]$$

where $k_1, k_2$ are the principle curvatures [1]. If we rotate the $n$ directions by $\frac{\pi}{2}$, we have

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i' &= \sum_{i=1}^{m} [k_1(\cos(\theta_i + \pi/2))^2 + k_2(\sin(\theta_i + \pi/2))^2] \\ &= \sum_{i=1}^{m} [k_1(\sin\theta_i)^2 + k_2(-\cos\theta_i)^2] \qquad (\cos(\theta + \pi/2) = \sin\theta, \sin(\theta + \pi/2) = -\cos\theta,)\end{aligned}$$

where $\lambda_1', ..., \lambda_m'$ are the corresponding normal curvatures after rotation. Then

$$\left(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \lambda_i'\right) = m(k_1 + k_2) = 2mH$$

and

$$\left(\sum_{i=1}^{m} \lambda_i - \sum_{i=1}^{m} \lambda_i'\right) = \sum_{i=1}^{m} (k_1 - k_2)[(\cos\theta_i)^2 - (\sin\theta_i)^2] = (k_1 - k_2)\sum_{i=1}^{m} \cos(2\theta_i)$$

Note that

$$\sum_{i=1}^{m} \cos(2\theta_i) = \sum_{i=1}^{m} \mathrm{Re}[e^{\mathbf{i}2\theta_i}] = \mathrm{Re}\left[\sum_{i=1}^{m} e^{\mathbf{i}2\theta_i}\right] = \mathrm{Re}[0] = 0$$

where $\mathbf{i}$ denotes the imaginary unit. Hence,

$$\lambda_i = \frac{1}{2}\left(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \lambda_i'\right) = mH$$

[1] $k_1$ is either maximum or minimum principle curvature

**Problem 4** *Show that a surface which is compact has an elliptic point.*

Let $C$ be a compact regular surface. Pick $c \in \mathbb{R}^3$. For any $x \in C$, the map $\phi(x) = ||x - c||$ is continuous, then it admits a maximum value, namely $r = \max_{x \in C} ||x - c||$. Let $p \in C \cap \mathcal{S}_r(c)$ on the intersection of $C$ and the sphere of radius $r$ centered at $c$.

Let $N$ be any plane containing $p$ that intersects $C$ at more than 1 point [2], then $N$ intersects $\mathcal{S}_r(c)$ at more than 1 point [3], then $N \cap \mathcal{S}_r(c)$ is a sphere of radius $0 < r_N \leq r$, then the curvature of $N \cap C$ is bounded below by a positive number, i.e. $\kappa_N \geq \frac{1}{r_N} \geq \frac{1}{r}$. Therefore, any principle curvature of $C$ at $p$ is bounded below by a positive number, then it must be an elliptic point or hyperbolic point.

Furthermore, $C$ is contained in one of the half space separated by $T_p\mathcal{S}_r(c)$, then $C$ cannot be a hyperbolic point.

**Problem 5** *Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$*

In the parameterization

$$x(u, v) = \left( \frac{\cos(u+v)}{\cos(u-v)}, \frac{\sin(u+v)}{\cos(u-v)}, \tan(u-v) \right)$$

Claim: asymptotic curves are the coordinate curves

Let $a = \cos(u+v), b = \sin(u+v), c = \cos(u-v), d = \sin(u-v)$, then we have

- $a^2 + b^2 = 1, c^2 + d^2 = 1$
- $ac - bd = \cos(2u), ac + bd = \cos(2v)$
- $ad + bc = \sin(2u), ad - bc = -\sin(2v)$
- $a_u = -b, a_v = -b$
- $b_u = a, b_v = a$
- $c_u = -d, c_v = d$
- $d_u = c, d_v = -c$

Calculation

$$\begin{aligned}
x &= \left( \frac{a}{c}, \frac{b}{c}, \frac{d}{c} \right) \\
x_u &= \left( \frac{ad - bc}{c^2}, \frac{ac + bd}{c^2}, \frac{1}{c^2} \right) \\
x_v &= \left( \frac{-ad - bc}{c^2}, \frac{ac - bd}{c^2}, \frac{-1}{c^2} \right) \\
N = x_u \times x_v &= \left( \frac{-2a}{c^3}, \frac{2b}{c^3}, \frac{2d}{c^3} \right) \\
N_u &= \left( \frac{2bc - 6ad}{c^4}, \frac{2ac + 6bd}{c^4}, \frac{2 + 4d^2}{c^4} \right) \\
N_v &= \left( \frac{2bc + 6ad}{c^4}, \frac{2ac - 6bd}{c^4}, \frac{-2 - 4d^2}{c^4} \right)
\end{aligned}$$

Second fundamental form

$$\begin{aligned}
e &= -\langle N_u, x_u \rangle = 0 \\
f &= -\langle N_u, x_v \rangle > 0 \\
g &= -\langle N_v, x_v \rangle = 0
\end{aligned}$$

Then the asymptotic curves are the coordinate curves w.r.t the parameterization

# 2 Appendix

[2] normal section guarantees the existence of $N$

[3] because $N$ intersects $C \subseteq \overline{\mathcal{B}_r(c)}$ at more than 1 point