# Chapter 2 HW

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## 3

If $[a', b']$ is a subinterval of $[a, b]$, show that $P[a', b'] \leq P[a, b]$ and $N[a', b'] \leq N[a, b]$

**Proof**

Let $\Gamma' = \{a' = x_0, x_1, ..., x_m = b'\}$ be any partition on $[a', b']$. Construct $\Gamma = \{a = x_{-1}, a' = a_0, x_1, ..., x_m = b', x_{m+1} = b\}$. Then,

$$\begin{aligned} P_\Gamma[f; a, b] &= \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})]^+ + [f(a') - f(a)]^+ + [f(b) - f(b')]^+ \\ &\geq P_{\Gamma'}[f; a', b'] \end{aligned}$$

Suppose, $P[a', b'] > P[a, b]$, i.e $\sup_{\Gamma'} P_{\Gamma'}[a', b'] > \sup_\Gamma P_\Gamma[a, b]$. Then there at least a $\Gamma_1'$ such that $P_{\Gamma_1'} > \sup_\Gamma P_\Gamma[a, b] \geq P_\Gamma$ for all $\Gamma$ on $[a, b]$. Contradiction

## 6

Let $f(x) = x^2 \sin \frac{1}{x}$ for $0 < x \leq 1$ and $f(0) = 0$. Show that $V[f; 0, 1] < +\infty$

**Proof**

As $f$ is continuous,

$$\begin{aligned} V[f; 0, 1] &= \int_0^1 |f'| dx \\ &= \int_0^1 \left| x \sin \frac{1}{x} - \cos \frac{1}{x} \right| dx \\ &< \int_0^1 2 dx \\ &= 2 \end{aligned}$$

## 9

Let $C$ be a curve with parametric equation $x = \Phi(t)$ and $y = \Psi(t)$, $a \leq t \leq b$

(a) If $\Phi$ and $\Psi$ are of bounded variation and continuous, show that $L = \lim_{|\Gamma| \to 0} l(\Gamma)$

(b) If $\Phi$ and $\Psi$ are continuously differentiable, show that $L = \int_a^b ([\Phi'(t)]^2 + [\Psi'(t)]^2)^{1/2}$

**Proof**

Let $\Gamma = \{a = t_0, t_1, ..., t_m = b\}$ be any partition on $[a, b]$

$$l(\Gamma) = \sum_{i=1}^{m} \sqrt{[\Phi(t_i) - \Phi(t_{i-1})]^2 + [\Psi(t_i) - \Psi(t_{i-1})]^2}$$

(a) $\Phi$ and $\Psi$ are of bounded variation, therefore,

$$\begin{aligned}
l(\Gamma) &= \sum_{i=1}^{m} \sqrt{[\Phi(t_i) - \Phi(t_{i-1})]^2 + [\Psi(t_i) - \Psi(t_{i-1})]^2} \\
&\leq \sum_{i=1}^{m} |\Phi(t_i) - \Phi(t_{i-1})| + \sum_{i=1}^{m} |\Psi(t_i) - \Psi(t_{i-1})| \\
&\leq \sup_{\Gamma} \sum_{i=1}^{m} |\Phi(t_i) - \Phi(t_{i-1})| + \sup_{\Gamma} \sum_{i=1}^{m} |\Psi(t_i) - \Psi(t_{i-1})| \\
&\leq V[\Phi, a, b] + V[\Psi, a, b] \\
&\leq +\infty
\end{aligned}$$

The set $\{l(\Gamma) : \Gamma\}$ is bounded above. Hence, $\sup_\Gamma l(\Gamma)$ exists

Given any $\epsilon > 0$, let $L = \sup_\Gamma l(\Gamma)$ and $\Gamma_1 = \{a = u_1, u_2, ..., u_{m_1} = b\}$ be a partition on $[a, b]$ such that $L - \frac{\epsilon}{2} < l(\Gamma_1) < L$.

$c = (\Phi, \Psi)$ is a continuous function on a compact set in a metric space into another metric space. By *Heine-Cantor Theorem*, $c$ is uniformly continuous, i.e for all $\frac{\epsilon}{4|\Gamma_1|} > 0$, there exists a $\eta > 0$ such that $||c(x) - c(y)|| < \frac{\epsilon}{4|\Gamma_1|}$ for all $x, y$ such that $|x - y| < \eta$

If $\Gamma_2 = \{a = v_1, v_2, ..., v_{m_1} = b\}$ is a partition on $[a, b]$ with $|\Gamma_2| < \min\{|\Gamma_1|, \eta\}$.

For every interval in $\Gamma_2$, there is at most one point from $\Gamma_1$, we write

$$l(\Gamma_2) = \sum_{i \in I} ||c(v_i) - c(v_{i-1})|| + \sum_{j \in J} ||c(v_j) - c(v_{j-1})||$$

where $J = \{j \in \{1, 2, ..., |\Gamma_2|\} : \exists k \in \{1, 2, ..., |\Gamma_1|\}, u_k \in [v_{j-1}, v_j]\}$ is the set of indices in $\Gamma_2$ such that the segment $[v_{j-1}, v_j]$ contains a point in $\Gamma_1$, namely $u_k$. $I = \{1, 2, ..., |\Gamma_2|\} \setminus J$

$$l(\Gamma_1 \cup \Gamma_2) = \sum_{i \in I} ||c(v_i) - c(v_{i-1})|| + \sum_{j \in J} ||c(u_k) - c(v_{j-1})|| + ||c(v_j) - c(u_k)||$$

By triangle inequality, $l(\Gamma_1 \cup \Gamma_2) \geq l(\Gamma_1)$

$$\begin{aligned}
l(\Gamma_2) &= \sum_{i \in I} ||c(v_i) - c(v_{i-1})|| + \sum_{j \in J} ||c(v_j) - c(v_{j-1})|| \\
&\geq \sum_{i \in I} ||c(v_i) - c(v_{i-1})|| \\
&= l(\Gamma_1 \cup \Gamma_2) - \sum_{j \in J} ||c(u_k) - c(v_{j-1})|| + ||c(v_j) - c(u_k)|| \\
&\geq l(\Gamma_1) - \sum_{j \in J} ||c(u_k) - c(v_{j-1})|| + ||c(v_j) - c(u_k)|| \\
&\geq l(\Gamma_1) - \sum_{j \in J} \frac{\epsilon}{4|\Gamma_1|} + \frac{\epsilon}{4|\Gamma_1|} && (v_{j-1} < u_k < v_j \text{ and } v_j - v_{j-1} < \eta) \\
&\geq l(\Gamma_1) - |\Gamma_1| \left( \frac{\epsilon}{4|\Gamma_1|} + \frac{\epsilon}{4|\Gamma_1|} \right) && (|J| = |\Gamma_1| - 2) \\
&= l(\Gamma_1) - \frac{\epsilon}{2} \\
&> L - \epsilon
\end{aligned}$$

Therefore, given any $\epsilon > 0$, construct $\Gamma_1$ and $\eta$, pick $\delta < \min\{|\Gamma_1|, \eta\}$, then $|L - l(\Gamma_2)| < \epsilon$ for all $\Gamma_2$ such that $|\Gamma_2| < \delta$

# 12

Prove that the construction of Theorem 2.30 is valid if the assumption that $\Phi$ is continuous is replaced by the assumption that $f$ and $\Phi$ has no common discontinuity.

**Proof**

# 15

Suppose that $f$ is continuous and $\Phi$ is of bounded variation on $[a, b]$. Show that the function $\Psi = \int_a^x f d\Phi$ is of bounded variation on $[a, b]$. If $g$ is continuous on $[a, b]$, show that $\int_a^b g d\Psi = \int_a^b g f d\Phi$

# 21

If $V[\phi; a, b] = +\infty$, show that there is a point $x_0 \in [a, b]$ such that either $V[\phi; I] = +\infty$ for every subinterval $I$ of $[a, b]$ having $x_0$ as left-hand end-point or $V[\phi; I] = +\infty$ for every subinterval $I$ of $[a, b]$ having $x_0$ as right-hand end-point.

## 24

Let $f$ be continuous and $\phi$ be bounded variation on $[a, b]$, and recall that the Riemann-Stieltjes integral $\int_a^b f d\phi$ then exists by Theorem 2.24. Show that $\lim_{\epsilon \to 0^+} \int_a^{a+\epsilon} f d\phi = 0$ if and only if either $f(a) = 0$ or $\phi$ is continuous at $a$. Deduce that the formula $\int_a^b f d\phi = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f d\phi$ may not hold

**Theorem 1 (Theorem 2.24)** *If $f$ is continuous on $[a, b]$ and $\phi$ is of bounded variation on $[a, b]$. Then $\int_a^b f d\phi$ exists. Moreover*

$$\left| \int_a^b f d\phi \right| \leq \sup |f[a, b]| V[\phi; a, b]$$

**Proof of ( $\Longrightarrow$ )**

We will prove that $\lim_{\epsilon \to 0^+} \int_a^{a+\epsilon} f d\phi = 0$ and $\phi$ is discontinuous at $a$ implies $f(a) = 0$.

Without loss of generality, assume $f(a) > 0$. Let $\epsilon > 0$, $\Gamma$ is a partition on $[a, a + \epsilon]$

$$R_\Gamma = \sum_{i=1}^m f(\xi_i)[\phi(x_i) - \phi(x_{i-1})]$$

As $f$ is continuous at $a$, for any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that $|f(x) - f(a)| < \epsilon_1$ for all $x \in (a, a + \delta_1)$. Let $\epsilon_1 = \frac{f(a)}{2}$, if $\epsilon < \delta_1$, we have

$$\begin{aligned} R_\Gamma &\geq \sum_{i=1}^m \frac{f(a)}{2}[\phi(x_i) - \phi(x_{i-1})] \\ &= \sum_{i=1}^m \frac{f(a)}{2}[\phi(x_i) - \phi(x_{i-1})] \end{aligned}$$

As $\phi$ is discontinuous at $a$, there exists $\epsilon_2 > 0$ such that for any $\delta_2 > 0$, there exists $x \in (a, a + \delta_2)$ such that $|\phi(x) - \phi(a)| > \epsilon_2$ that implies $V[\phi; a, a + \epsilon] \geq \epsilon_2 > 0$ for all $\epsilon > 0$. Therefore, the monotone function $V[\phi; a, a + \epsilon]$ has limit $\lim_{\epsilon \to 0+} V[\phi; a, a + \epsilon] \geq \epsilon_2 > 0$

## 27

If $f$ is an even function on $[-1, 1]$, verify the formula $V[f; -1, 1] = 2P[f; -1, 1]$, $V[f; 0, 1] = P[f; -1, 1]$ and $P[f; 0, 1] = N[f; -1, 0]$

**Proof**

Suppose $f$ is bounded variation on $[-1, 1]$. Let $\Gamma_1$ be any partition on $[-1, 1]$. We define $\Gamma_2 = \{-x : x \in \Gamma_1\}$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. $\Gamma$ is of the form, namely *symmetric partition*

$$\Gamma = \{-1 = -x_m, ..., -x_1, x_0 = 0, x_1, ..., x_m = 1\}$$

where $x_1$ might or might not equal 0. As $\Gamma$ is a refinement of $\Gamma_1$, we immediately have

$$\begin{aligned} P_{\Gamma_1} &\leq P_\Gamma \\ N_{\Gamma_1} &\leq N_\Gamma \\ S_{\Gamma_1} &\leq S_\Gamma \end{aligned}$$

That is,

$$\begin{aligned} P[a,b] &= \sup_{\Gamma_1} P_{\Gamma_1}[a,b] = \sup_{\Gamma \text{ is symmetric}} P_\Gamma[a,b] \\ N[a,b] &= \sup_{\Gamma_1} N_{\Gamma_1}[a,b] = \sup_{\Gamma \text{ is symmetric}} P_\Gamma[a,b] \\ V[a,b] &= \sup_{\Gamma_1} S_{\Gamma_1}[a,b] = \sup_{\Gamma \text{ is symmetric}} S_\Gamma[a,b] \end{aligned}$$

**Proof for** $P[-1,1] = V[0,1]$
Let $\Gamma$ be a symmetric partition on $[-1,1]$

$$\begin{aligned} P_\Gamma[-1,1] &= \sum_{i=1}^{m}[f(x_i) - f(x_{i-1})]^+ + \sum_{i=1}^{m}[f(-x_{i-1}) - f(-x_i)]^+ \\ &= \sum_{i=1}^{m}[f(x_i) - f(x_{i-1})]^+ + \sum_{i=1}^{m}[f(x_{i-1}) - f(x_i)]^+ \qquad (f \text{ is even}) \\ &= \sum_{i=1}^{m}[f(x_i) - f(x_{i-1})]^+ + [f(x_{i-1}) - f(x_i)]^+ \\ &= \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \\ &= S_\Gamma[0,1] \end{aligned}$$

Therefore, the two sets $\{P_\Gamma[-1,1] : \Gamma \text{ is symmetric}\}$ and $\{S_\Gamma[0,1] : \Gamma \text{ is symmetric}\}$ are identical. So $P[-1,1] = V[0,1]$

**Proof for** $2P[-1,1] = V[-1,1]$

Let $\Gamma$ be a symmetric partition on $[-1,1]$

$$\begin{aligned}
P_\Gamma[-1,1] &= \sum_{i=1}^{m}[f(x_i)-f(x_{i-1})]^+ + \sum_{i=1}^{m}[f(-x_{i-1})-f(-x_i)]^+ \\
&= \sum_{i=1}^{m}[f(-x_i)-f(-x_{i-1})]^+ + \sum_{i=1}^{m}[f(-x_{i-1})-f(-x_i)]^+ \quad (f \text{ is even}) \\
&= \sum_{i=1}^{m}[f(-x_i)-f(-x_{i-1})]^+ + [f(-x_{i-1})-f(-x_i)]^+ \\
&= \sum_{i=1}^{m}|f(-x_{i-1})-f(-x_i)| \\
&= S_\Gamma[-1,0]
\end{aligned}$$

Similarly, $P[-1,1]=V[-1,0]$. By linearity of variation, $S_\Gamma[-1,1]=S_\Gamma[-1,0]+S_\Gamma[0,1]=2P_\Gamma[-1,1]$

**Proof for** $P[0,1]=N[-1,0]$

Let $\Gamma$ be a symmetric partition on $[-1,1]$

$$\begin{aligned}
P_\Gamma[0,1] &= \sum_{i=1}^{m}[f(x_i)-f(x_{i-1})]^+ \\
&= \sum_{i=1}^{m}[f(x_{i-1})-f(x_i)]^- \\
&= \sum_{i=1}^{m}[f(-x_i)-f(-x_{i-1})]^- (f \text{ is even}) \\
&= N_\Gamma[-1,0]
\end{aligned}$$

Using the same argument, $P[0,1]=N[-1,0]$