ma5209 assignment 3

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1 Problem 1

(a) Let p be a prime number and let $\mathbb{Z}[1/p]$ be the subring of $\mathbb{Q}$ consisting of rational numbers whose denominators are powers of p. Construct compatible maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}[1/p]/\mathbb{Z}$ and show that

$$\varinjlim \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}[1/p]/\mathbb{Z}$$

This is the "p-torsion Prüfer group"

(b) Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$ where the direct sum runs over the prime numbers.

(c) Show that $\mathrm{Tor}_1(A, \mathbb{Z}_{p^\infty})$ and $\mathrm{Tor}_1(A, \mathbb{Q}/\mathbb{Z})$ are naturally isomorphic to certain subgroups of A

(d) Compute $H_*(\mathbb{R}P^n; \mathbb{Q}/\mathbb{Z})$

1.1 (a)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p^1\mathbb{Z} \arrow[r] & \dots \arrow[r] & \mathbb{Z}/p^{n-1}\mathbb{Z} \arrow[r, "f_n"] \arrow[dr, "g_{n-1}"'] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d, "g_n"] & \dots \\
& & & & \mathbb{Z}[1/p]/\mathbb{Z} &
\end{tikzcd}$$

We define $f_n : \mathbb{Z}/p^{n-1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ and $g_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}[1/p]/\mathbb{Z}$ as follows:

$$f_n : [a] \mapsto [pa]$$

$$g_n : [a] \mapsto \left[\frac{a}{p^n}\right]$$

where $a \in \mathbb{Z}$. We will verify that f_n, g_n are well-defined, group homomorphisms and $\mathbb{Z}[1/p]/\mathbb{Z}$ is the direct limit.

1. f_n is well-defined
$$f_n([a + kp^{n-1}]) = [p(a + kp^{n-1})] = [pa + kp^n] = [pa] = f([a])$$

2. f_n is a homomorphism
$$f_n([a] + [b]) = f_n([a + b]) = [p(a + b)] = [pa + pb] = [pa] + [pb] = f_n([a]) + f_n([b])$$

3. g_n is well-defined
$$g_n([a + kp^n]) = \left[\frac{a + kp^n}{p^n}\right] = \left[\frac{a}{p^n} + k\right] = \left[\frac{a}{p^n}\right] = g_n([a])$$

4. g_n is a homomorphism
$$g_n([a] + [b]) = g_n([a + b]) = \left[\frac{a + b}{p^n}\right] = \left[\frac{a}{p^n} + \frac{b}{p^n}\right] = \left[\frac{a}{p^n}\right] + \left[\frac{b}{p^n}\right] = g_n([a]) + g_n([b])$$

Note that $\left[\frac{a}{p^n} + \frac{b}{p^n}\right] = \left[\frac{a}{p^n}\right] + \left[\frac{b}{p^n}\right]$ is due to x and $x + 1$ identify the same element in p-torsion Prüfer group

5. $g_{n-1} = g_n f_n$

$$g_n(f_n([a])) = g_n([pa]) = \left[\frac{pa}{p^n}\right] = \left[\frac{a}{p^{n-1}}\right] = g_{n-1}([a])$$

6. direct limit

   Note that each f_n and g_n is a monomorphism, we have the filtration

$$0 \subseteq g_1(\mathbb{Z}/p^1\mathbb{Z}) \subseteq ... \subseteq g_{n-1}(\mathbb{Z}/p^{n-1}\mathbb{Z}) \subseteq g_n(\mathbb{Z}/p^n\mathbb{Z}) \subseteq ... \subseteq \mathbb{Z}[1/p]/\mathbb{Z}$$

   and $\mathbb{Z}[1/p]/\mathbb{Z} = \bigcup_{n=0}^{\infty} g_n(\mathbb{Z}/p^n\mathbb{Z})$ which is exactly the direct limit.

1.2 (b)

Define an isomorphism $\alpha : \bigoplus_p \mathbb{Z}_{p^\infty} \to \mathbb{Q}/\mathbb{Z}$ as follows

$$\alpha : \left(\left[\frac{a_1}{p_1^{n_1}}\right], \left[\frac{a_2}{p_2^{n_2}}\right], ...\right) \mapsto \left[\sum_{i=1}^{\infty} \frac{a_i}{p_i^{n_i}}\right]$$

We will prove that α is an isomorphism by verifying α is a bijective homomorphism

1. α is an homomorphism

   α is a direct product of inclusion maps

2. α is injective, $\ker \alpha = \{0\}$

   Let $\left(\left[\frac{a_1}{p_1^{n_1}}\right], \left[\frac{a_2}{p_2^{n_2}}\right], ...\right)$ is mapped into $0 \in \mathbb{Q}/\mathbb{Z}$, that is

$$0 = \left[\sum \frac{a_i}{p_i^{n_i}}\right] = \left[\frac{\sum a_i \left(\prod_{j \neq i} p_j^{n_j}\right)}{\prod p_i^{n_i}}\right]$$

   Then

$$\sum a_i \left(\prod_{j \neq i} p_j^{n_j}\right) - k\left(\prod p_i^{n_i}\right) = 0$$

   In $\pmod{p_i^{n_i}}$, we have

$$a_i \pmod{p_i^{n_i}} = 0$$

   That is,

$$\left(\left[\frac{a_1}{p_1^{n_1}}\right], \left[\frac{a_2}{p_2^{n_2}}\right], ...\right) = (0, 0, ...) = 0$$

3. α is surjective

   Lemma 1. *Given p, q are coprime, if $0 \leq m < pq$, then there is a decomposition*

$$\frac{m}{pq} = \frac{a}{p} + \frac{b}{q}$$

   More generally, given $p_1, p_2, ...$ are primes, if $0 \leq m < p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, then there is a decomposition

$$\frac{m}{p_1^{n_1} p_2^{n_2} ... p_k^{n_k}} = \sum_{i=1}^{k} \frac{a_i}{p_i^{n_i}}$$

   α being surjective is directly from lemma **??**. Lemma **??** is done as follows: given p, q coprime, there exists a_1, b_1 such that $a_1 p + b_1 q = 1$, construct $a = ma_1, b = mb_1$

1.3 (c)

Lemma 2. Tor *is symmetric*

$$\mathrm{Tor}_i(A,B) \cong \mathrm{Tor}_i(B,A)$$

Lemma 3. Tor *commutes with direct limit*

$$\mathrm{Tor}_i(\varinjlim_{\alpha} A_\alpha, B) \cong \varinjlim_{\alpha} \mathrm{Tor}_i(A_\alpha, B)$$

Lemma 4. $\mathbb{Q}/\mathbb{Z}$ *can be written as a direct limit*

$$\mathbb{Q}/\mathbb{Z} = \varinjlim_{n} h_n(\mathbb{Z}/n)$$

where $h_n : z \mapsto z/n$ *and each* $h_n(\mathbb{Z}/n) \cong \mathbb{Z}/n$

Lemma 5. *For any abelian group* N*,*

$$\mathrm{Tor}_1(\mathbb{Z}/n, N) = \ker(n : N \to N) = \{x \in N : nx = 0\} = nN$$

1.3.1 $\mathbb{Z}_{p^\infty}$

$$\begin{aligned}
\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, A) &= \mathrm{Tor}_1^{\mathbb{Z}}(\varinjlim_{n} g_n(\mathbb{Z}/p^n), A)\\
&= \varinjlim_{n} \mathrm{Tor}_1^{\mathbb{Z}}(g_n(\mathbb{Z}/p^n), A)\\
&= \varinjlim_{n} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^n, A)\\
&= \varinjlim_{n} \{x \in A : p^n x = 0\}\\
&= \{x \in A : \exists n \in \mathbb{N}, p^n x = 0\} \trianglelefteq A
\end{aligned}$$

1.3.2 $\mathbb{Q}/\mathbb{Z}$

$$\begin{aligned}
\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) &= \mathrm{Tor}_1^{\mathbb{Z}}(\varinjlim_{n} h_n(\mathbb{Z}/n), A)\\
&= \varinjlim_{n} \mathrm{Tor}_1^{\mathbb{Z}}(h_n(\mathbb{Z}/n), A)\\
&= \varinjlim_{n} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A)\\
&= \varinjlim_{n} \{x \in A : nx = 0\}\\
&= \{x \in A : \exists n \in \mathbb{N}, nx = 0\} \trianglelefteq A
\end{aligned}$$

1.4 (d)

Recall the homology of $\mathbb{R}P^n$ with abelian group A coefficients

Proposition 1.

$$\begin{aligned}
&H_0(\mathbb{R}P^n, A) = A\\
&H_q(\mathbb{R}P^n, A) = \begin{cases} A/2A & 1 \le q < n,\ q \text{ odd} \\ {}_2A & 1 \le q < n,\ q \text{ even} \end{cases}\\
&H_n(\mathbb{R}P^n, A) = \begin{cases} A & n \text{ odd} \\ {}_2A & n \text{ even} \end{cases}\\
&H_q(\mathbb{R}P^n, A) = 0 \qquad\qquad n < q
\end{aligned}$$

where ${}_2A = \ker(2 : A \to A)$

Let $A = \mathbb{Q}/\mathbb{Z}$, then

- $2A = \mathbb{Q}/\mathbb{Z}$, $A/2A = \mathbb{Q}/\mathbb{Z}$
- ${}_2A = \ker(2 : A \to A) = \{[n/2] : n \in \mathbb{Z}\}$

[1]

2 Problem 2

Let $I_\bullet$ denote the chain complex with $I_0 = \mathbb{Z} \oplus \mathbb{Z}, I_1 = \mathbb{Z}, I_q = 0$ for $q \neq 0, 1$ and $\partial : I_1 \to I_0$ given by $1 \mapsto (+1, -1)$. Show that it is isomorphic to the chain complex of a CW structure on the unit interval. Let $C_\bullet$ and $D_\bullet$ be chain complexes. Show that there is a bijective correspondence between triples (f_0, f_1, h) where $f_0, f_1 : C_\bullet \to D_\bullet$ are chain maps and h is a chain homotopy from f_0 to f_1 and chain maps $C_\bullet \otimes I_\bullet \to D_\bullet$

2.1 CW structure on $I_\bullet$

Define CW structure $X_0 \subseteq X_1 = X_2 = ... = X$ as follows

- X_0 contains two points: $C_0^{CW}(X) = \mathbb{Z} \oplus \mathbb{Z}$
- $X_1 = I$: $a_1^{(1)} : S_1^0 \mapsto X_0$ maps two points of S_1^0 to two points of X_0, $C_1^{CW}(X) = \mathbb{Z}$

$$\begin{tikzcd} S_1^0 \arrow[r] \arrow[d, "a_\bullet^{(1)}"] & D_1^1 \arrow[d, "c_\bullet^{(1)}"] \\ X_0 \arrow[r] & X_1 \cong I \end{tikzcd}$$

We will verify that $d_0 : C_1^{CW}(X) \to C_0^{CW}(X)$ is ∂. Indeed, as $X_0 \subseteq X_1$, we have the short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(X_0) \longrightarrow C_\bullet(X_1) \longrightarrow C_\bullet(X_1, X_0) \longrightarrow 0$$

which induces

$$\begin{tikzcd} & H_1(X_1) \arrow[r] & H_1(X_1, X_0) = C_1^{CW}(X) \arrow[dll, "d"'] \\ H_0(X_0) = C_0^{CW}(X) & & \end{tikzcd}$$

The map $d : C_1^{CW}(X) \to C_0^{CW}(X)$ is defined as follows

- Choose a generator of $H_1(X_1, X_0)$ that is a non-zero chain c in $C_1(X_1, X_0)$ such that $\partial c = 0$: Let $c : \Delta^1 \to X_1$ such that $c(0)$ and $c(1)$ are the two points of X_0, then image ∂c is in X_0 which is zero in $C_0(X_1, X_0)$
- Let $b \in C_1(X_1)$, $b = c$
- Let $a \in C_0(X_0)$ such that $a = \partial b$. Then $a = +x_1 - x_2$ where x_1, x_2 are the two singular 0-simplex in X_0

Hence, $d : 1 \mapsto (+1, -1)$

2.2 Bijective correspondence between (f_0, f_1, h) and chain maps $C_\bullet \otimes I_\bullet \to D_\bullet$

We decompose $(C_\bullet \otimes D_\bullet)_n$ as follows

$$\begin{aligned} (C_\bullet \otimes D_\bullet)_n &= \bigoplus_{p+q=n} C_p \otimes I_q \\ &= (C_n \otimes I_0) \oplus (C_{n-1} \otimes I_1) \end{aligned}$$

[1]not sure if this question requires students to use Universal Coefficient Theorem to convert homology over $\mathbb{Z}$ to homology over $\mathbb{Q}/\mathbb{Z}$

2.2.1 $(C_\bullet \otimes I_\bullet \to D_\bullet) \mapsto (f_0, f_1, h)$

Let $H : C_\bullet \otimes I_\bullet \to D_\bullet$ be a chain map that is a sequence of maps [2]

$$
\begin{aligned}
H_n &: (C_n \otimes I_0) \oplus (C_{n-1} \otimes I_1) \to D_n \\
H_{n-1} &: (C_{n-1} \otimes I_0) \oplus (C_{n-2} \otimes I_1) \to D_{n-1}
\end{aligned}
$$

Define the following (we use $a \oplus b$ for an element of $A \oplus B$ where $a \in A, b \in B$)

$$
\begin{aligned}
f_0 : C_n &\to D_n \\
x &\mapsto H((x \otimes (1,0)) \oplus (0 \otimes 0)) \\
f_1 : C_n &\to D_n \\
x &\mapsto H((x \otimes (0,1)) \oplus (0 \otimes 0)) \\
h : C_{n-1} &\to D_n \\
y &\mapsto s(n-1)H((0 \otimes (0,0)) \oplus (y \otimes 1))
\end{aligned}
$$

where $a, b, c \in \mathbb{Z}$ and $s(n)$ is the sign function defined in the boundary map of tensor product of chain complexes. Let $(x \otimes (a,b)) \oplus (y \otimes c) \in (C_n \otimes I_0) \oplus (C_{n-1} \otimes I_1)$, we can write H in terms of f_0, f_1, h as follows:

$$H((x \otimes (a,b)) \oplus (y \otimes c)) = af_0(x) + bf_1(x) + s(n-1)ch(y)$$

Furthermore,

$$
\begin{aligned}
\partial H((x \otimes (a,b)) \oplus (y \otimes c)) &= \partial(af_0(x) + bf_1(x) + s(n-1)ch(y)) \\
&= a\partial f_0(x) + b\partial f_1(x) + s(n-1)c\partial h(y)
\end{aligned}
$$

$$
\begin{aligned}
H\partial((x \otimes (a,b)) \oplus (y \otimes c)) &= H((\partial x \otimes (a,b) + s(n-1)y \otimes (+c, -c)) \oplus (\partial y \otimes c)) \\
&= af_0(\partial x) + bf_1(\partial x) + s(n-1)cf_0(y) - s(n-1)cf_1(y) + s(n-2)ch(\partial y)
\end{aligned}
$$

Since H, f_0, f_1 are a chain maps, $H\partial = \partial H$, let

- $a = 1, b = 0, c = 0$, then $f_0\partial = \partial f_0$
- $a = 0, b = 1, c = 0$, then $f_1\partial = \partial f_1$
- $a = 0, b = 0, c = 1$, then $\partial h = f_0 - f_1 + h\partial$ (because $s(n-2) = -s(n-1)$)

$$
\begin{tikzcd}
\partial x \otimes (a,b) \in C_{n-1} \otimes I_0 \arrow[d, "H_{n-1}"] & x \otimes (a,b) \in C_n \otimes I_0 \arrow[l, "\partial"'] \arrow[dd, "H_n"] \\
af_0(\partial x) + bf_1(\partial y) & \\
a\partial f_0(x) + b\partial f_1(\partial y) & af_0(x) + bf_1(x) \arrow[l, "\partial"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\partial y \otimes c) \oplus (s(n-1)y \otimes (+c, -c)) \in C_{n-2} \otimes I_0 \oplus C_{n-1} \otimes I_0 \arrow[d, "H_{n-1}"] & y \otimes c \in C_{n-1} \otimes I_1 \arrow[l, "\partial"'] \arrow[dd, "H_n"] \\
s(n-2)ch(\partial y) + s(n-1)f_0(y) - s(n-1)f_1(y) & \\
s(n-1)c\partial h(y) & s(n-1)ch(y) \arrow[l, "\partial"]
\end{tikzcd}
$$

[3] [4]

[2] sorry for picking the symbol H that looks like homology

[3] apparently this is related to the concept of inner chain

[4] `https://mathoverflow.net/questions/59357/why-chain-homotopy-when-there-is-no-topology-in-the-background`

2.2.2 $\quad (f_0, f_1, h) \mapsto (C_\bullet \otimes I_\bullet \to D_\bullet)$

In the previous argument, we can write H in terms of three maps (f_0, f_1, h). Moreover, $H\partial = \partial H$ is a consequence of f_0, f_1 being chain maps and h being the homotopy from f_0 to f_1

3 Problem 3

Prove that the Universal Coefficient Theorem short exact sequence splits (though this splitting cannot be made natural)

Theorem 1 (universal coefficient theorem)**.** *Let R be a PID and N be an R-module. For any chain complex of free R-modules $C_\bullet$ there is a short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes_R N \xrightarrow{\alpha} H_n(C_\bullet \otimes_R N) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

where the natural transformation α is defined by the bilinear map

$$\begin{gathered} H_n(C_\bullet) \times N \to H_n(C_\bullet \otimes N) \\ ([c], n) \mapsto [c \otimes_R n] \end{gathered}$$

where $c \in C_\bullet, n \in N$

Lemma 6. *Given the short exact sequence of R-modules*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 0$$

if C is free, then the sequence splits.

Lemma 7 (freedom theorem for modules over a PID)**.** *If R is a PID, any submodule of a free R-module is free*

3.1 Proof

Let $Z_n = \ker(\partial : C_n \to C_{n-1}), B_n = \mathrm{im}(\partial : C_{n+1} \to C_n)$. Z_n and B_n are chain complexes (with boundary map being the zero map), then we have the exact sequence of chain complexes (the inclusion i and boundary ∂ are both chain maps)

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

As B_n is a submodule of free R-module C_n, B_n is free, hence the sequence splits. That is, there exists a chain map $p : C_n \to Z_n$ which extends to the map $f : C_n \to H_n(C_\bullet)$ (note that the quotient map $Z_n \to H_n(C_\bullet)$ is also a chain map)

$$\begin{tikzcd} Z_n \arrow[d, two heads] & C_n \arrow[l, "p"'] \arrow[dl, dashed, "f"] \\ H_n(C_\bullet) = Z_n/B_n & \end{tikzcd}$$

$$\begin{tikzcd} \cdots & C_{n-1} \arrow[l] \arrow[d, "f"] & C_n \arrow[l, "\partial"'] \arrow[d, "f"] & C_{n+1} \arrow[l, "\partial"'] \arrow[d, "f"] & \cdots \arrow[l] \\ \cdots & H_{n-1}(C_\bullet) \arrow[l] & H_n(C_\bullet) \arrow[l, "0"'] & H_{n+1}(C_\bullet) \arrow[l, "0"'] & \cdots \arrow[l] \end{tikzcd}$$

Let F : R-Mod $\to$ R-Mod be defined by $F(X) = X \oplus_R N$. F is additive, that extends to a functor Ch(R-Mod) $\to$ Ch(R-Mod)

$$\begin{tikzcd} \cdots & C_{n-1} \otimes_R N \arrow[l] \arrow[d, "Ff"] & C_n \otimes_R N \arrow[l, "F\partial"'] \arrow[d, "Ff"] & C_{n+1} \otimes_R N \arrow[l, "F\partial"'] \arrow[d, "Ff"] & \cdots \arrow[l] \\ \cdots & H_{n-1}(C_\bullet) \otimes_R N \arrow[l] & H_n(C_\bullet) \otimes_R N \arrow[l, "0"'] & H_{n+1}(C_\bullet) \otimes_R N \arrow[l, "0"'] & \cdots \arrow[l] \end{tikzcd}$$

Take the homology of top chain and bottom chain, note that the boundary map in the bottom chain is the zero map, then $H_n(H_\bullet(C_\bullet) \otimes_R N) = H_n(C_\bullet) \otimes_R N$. We have the map $H_n(Ff)$

$$\begin{tikzcd} H_n(C_\bullet) \otimes_R N \arrow[r, "\alpha"] \arrow[rr, dashed, bend right, "1"'] & H_n(C_\bullet \otimes N) \arrow[r, "H_n(Ff)"] & H_n(C_\bullet) \otimes_R N \end{tikzcd}$$

We will verify that $H_n(Ff)\alpha = 1$. Let $[c] \in H_n(C_\bullet)$ where $c \in \mathbb{Z}_n$ and $n \in N$

$$\alpha : [c] \otimes_R n \mapsto [c \otimes_R n]$$

On the other hand, Ff is defined by

$$Ff : c \otimes_R n \mapsto [c] \otimes_R n$$

Hence,

$$H_n(Ff) : [c \otimes_R n] \mapsto [c] \otimes_R n$$

$$\begin{tikzcd}
H_n(C_\bullet) \otimes_R N \arrow[r, dashed, "\alpha"] & H_n(C_\bullet \otimes N) \arrow[r, dashed, "H_n(Ff)"] & H_n(C_\bullet) \otimes_R N \\
H_n(C_\bullet) \times N \arrow[u, "{([c],n) \mapsto [c] \otimes_R n}"] \arrow[ur, "{([c],n) \mapsto [c \otimes_R n]}"'] & C_n \otimes N \arrow[u, two heads, "c \otimes_R n \mapsto [c \otimes_R n]"] \arrow[ur, "c \otimes_R n \mapsto [c] \otimes_R n"']
\end{tikzcd}$$

4 Problem 4

The Eilenberg-Zilber map

$$EZ : \bigoplus_{p+q=n} C_p(X) \times C_q(Y) \to C_n(X \times Y)$$

is defined on the (p, q) summand to be linear map defined by the bilinear map sending (σ, τ) to

$$\sum_\gamma (-1)^{A(\gamma)} (\sigma \times \tau)\gamma$$

where γ runs over injective affine maps $\gamma : \Delta^n \to \Delta^p \times \Delta^q$ which sends vertices to pairs of vertices and which are such that each of the projections to the two factor are order-preserving on vertices. Such that a map γ traces out a staircase in the plane, running from $(0, 0)$ to (p, q) and $A(\gamma)$ is the area under the staircase

Show that when $q = 1$ this is just the "prism" operator we used to show that homology is a homotopy invariant.

Show that EZ is a chain map

4.1 $q = 1$

Let $t \in [n] = \{0, 1, ..., n\}$ denote the vertex of Δ^n simplex, $p(t), q(t)$ denote the corresponding vertices of Δ^p, Δ^q. The functions $p(t), q(t)$ are monotone increasing and $(p(0), q(0)) = (0, 0)$, $(p(n), q(n)) = (p, q)$. When $q = 1$, there exists $t_1 \in [n-1]$ such that $p(t_1) = p(t_1 + 1)$ and $q(t_1) = 0, q(t_1 + 1) = 1$. Hence,

$$(p(t), q(t)) = \begin{cases} (t, 0) & \text{if } 0 \leq t \leq t_1 \\ (t, 1) & \text{if } t_1 + 1 \leq t \leq n \end{cases}$$

Let γ defined from $p(t), q(t)$, then $A(\gamma) = p - t_1$. This is exactly the definition of prism operator when we choose the appropriate vertices of I.

4.2 EZ is a chain map

Let $x_p \otimes y_q \in C_p(X) \otimes C_q(Y)$, then

$$\begin{aligned} EZ\partial(x_p \otimes y_q) &= EZ(\partial x_p \otimes y_q \oplus s(p)x_p \otimes \partial y_q) \\ &= EZ(\partial x_p \otimes y_q) + EZ(s(p)x_p \otimes \partial y_q) \end{aligned}$$

$$EZ(x_p \otimes y_q) = \sum_{\gamma \in (p,q)} (-1)^{A(\gamma)} (x_p \times y_q)\gamma$$

where $\gamma \in (p, q)$ denotes all staircases from $(0, 0) \to (p, q)$. We will show that $\partial EZ(x_p \otimes y_q) = EZ\partial(x_p \otimes y_q)$. For each $\gamma : \Delta^n \to \Delta^p \times \Delta^q \in (p, q)$, write γ as

$$\gamma = (l_0, l_1, ..., l_n) = ((p_i, q_i))_{i=0}^n = ((0, 0), ..., (p, q))$$

consider the boundary of $(-1)^{A(\gamma)}(x_p \times y_q)\gamma$

$$\partial EZ(x_p \otimes y_q) = \sum_{\gamma \in (p,q)} \partial((-1)^{A(\gamma)}(x_p \times y_q)\gamma) = \sum_{\gamma \in (p,q)} \sum_{i=0}^{n} (-1)^i (-1)^{A(\gamma)} (x_p \times y_q)\gamma d^i$$

where $d^i : \Delta^{n-1} \to \Delta^n$ is the face map. There are 4 cases of removing vertex i as in Figure **??**

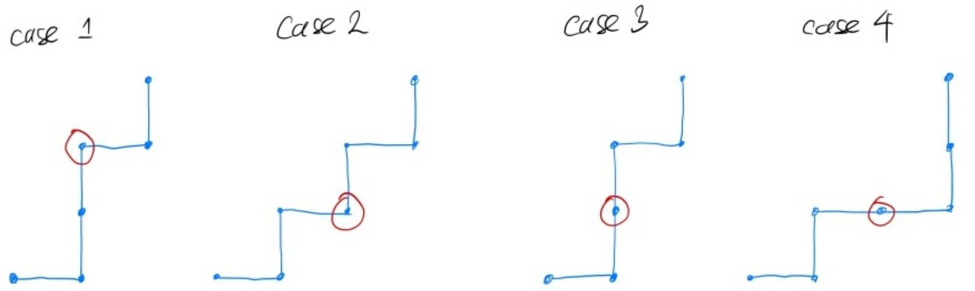

Figure 1: 4 staircases, note that if removing vertex is at the beginning or the end of staircase, it falls into these 4 cases by shifting indices of $\gamma, \Delta^p, \Delta^q$ by 2 since shifting by 2 does not change the sign of boundary map of singular simplex and sign of Eilenberg-Zilber map

4.2.1 Case 1, 2

There is a bijection of (γ, i) of case 1 and (γ', i') of case 2 ($i = i'$) such that the composition γd^i and $\gamma' d^{i'}$ result in the same map $\Delta^{n-1} \to \Delta^p \times \Delta^q$. Moreover, as the area under the staircase of γ and γ' differ by one unit square, $A(\gamma) = -A(\gamma')$. Hence, all pairs (γ, i), (γ', i') cancel out.

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d^i"] \arrow[rr, bend right, "\gamma d^i"'] & \Delta^n \arrow[r, "\gamma"] & \Delta^p \times \Delta^q \end{tikzcd}$$

4.2.2 Case 4

Let the removing vertex be $l_i = (p_i, q_i)$. Note that, $q_{i-1} = q_i = q_{i+1}$, then (l_{i-1}, l_i, l_{i+1}) is a triangle on the projection of $\Delta^p \times \Delta^q$ on the subspace $q_\bullet = q_i$. By removing l_i, $(x_p \times y_q)\gamma d^i$ corresponds to one of the map in $EZ(\partial x_p \otimes y_q)$, specifically

$$\begin{aligned} EZ(\partial x_p \otimes y_q) &= \sum_{\gamma_1 \in (p-1,q)} (-1)^{A(\gamma_1)} (\partial x_p \times y_q)\gamma_1 \\ &= \sum_{\gamma_1 \in (p-1,q)} (-1)^{A(\gamma_1)} \left(\left(\sum_{j=0}^{p} (-1)^j x_p d^j \right) \times y_q \right) \gamma_1 \\ &= \sum_{\gamma_1 \in (p-1,q)} \sum_{j=1}^{p} (-1)^{A(\gamma_1)} (-1)^j (x_p \times x_q)(d^j \times 1)\gamma_1 \end{aligned}$$

$$\Delta^{p-1} \xrightarrow{\quad d^j \quad} \Delta^p$$

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "\gamma_1"] \arrow[rr, bend right, "(d^j \times 1)\gamma_1"'] & \Delta^{p-1} \times \Delta^q \arrow[r, "d^j \times 1"] & \Delta^p \times \Delta^q \end{tikzcd}$$

We will construct a bijection between case 3 and $EZ(\partial x_p \otimes y_q)$. The equality holds if we pick $j = p_i$ and γ_1 is γ with l_i removed.

$$(-1)^i (-1)^{A(\gamma)} (x_p \times x_q)\gamma d^i = (-1)^{A(\gamma_1)} (-1)^j (x_p \times x_q)(d^j \times 1)\gamma_1$$

By removing $l_i = (p_i, q_i)$, the map γd^i factors through $(d^j \times 1)\gamma_1$, that is, $(x_p \times x_q)\gamma d^i = (x_p \times x_q)(d^j \times 1)\gamma_1$. Furthermore, we have two equalities $i = p_i + q_i$ and $A(\gamma) = q_i A(\gamma_1)$, then $(-1)^i(-1)^{A(\gamma)} = (-1)^{A(\gamma_1)}(-1)^j$. Note that, this is indeed a bijection since the only case the map γd^i factors through $(d^j \times 1)\gamma_1$ is when p_i appears once in the sequence $(l_1, l_2, ..., l_n)$. Figure **??** is an illustration when $p = 3, q = 2, q_i = 1$

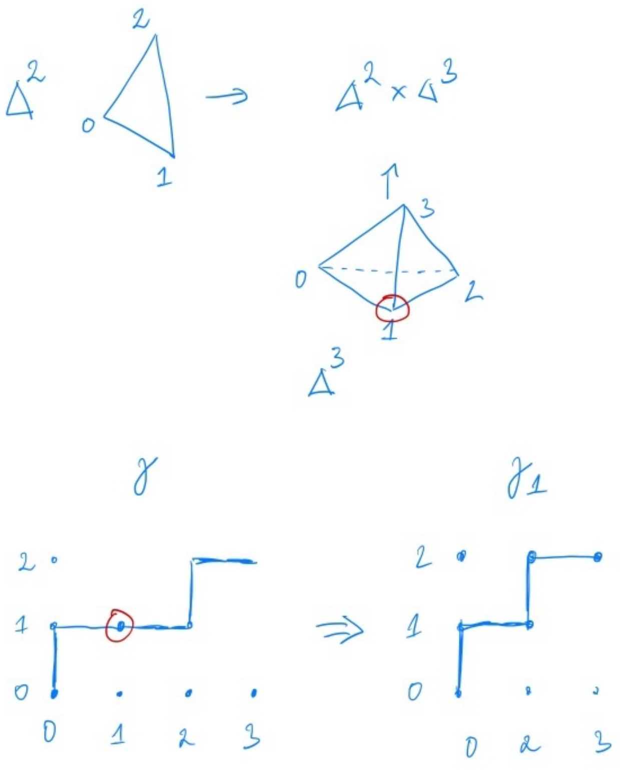

Figure 2: staircase removed

4.2.3 Case 3

Case 3 is similar to case 4, the bijection is on $EZ(x_p \otimes \partial y_q)$

5 Problem 5

Let K denote the Klein bottle, obtained by gluing two Möbius strips together along their boundaries. Compute the graded abelian group $H_*(K \times K \times K)$

5.1 Homology of K

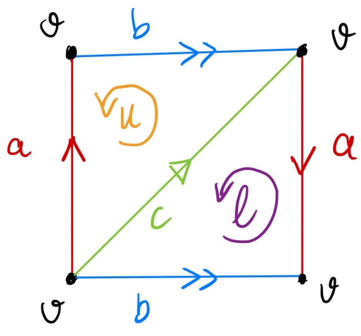

Figure 3: Klein bottle

Denote the Δ-complex diagram as in Figure **??**,

$$
\begin{array}{ccccccccccc}
dim: & & 0 & & 1 & & 2 & & 3 & & ... \\
0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\partial_1} & 3\mathbb{Z} & \xleftarrow{\partial_2} & 2\mathbb{Z} & \longleftarrow & 0 & & ...
\end{array}
$$

Then, the map ∂_1 and ∂_2 are defined as follows

$$
\begin{aligned}
\partial_1 : C_1 &\to C_0 \\
a &\mapsto 0 \\
b &\mapsto 0 \\
c &\mapsto 0 \\
\partial_2 : C_2 &\to C_1 \\
u &\mapsto -a-b+c \\
l &\mapsto -a+b-c
\end{aligned}
$$

As $\partial_1 = 0$, then

$$H_0(K) = \mathbb{Z}$$

Let change of basis as follows

$$
\begin{aligned}
a' &= -a+b-c \\
b' &= b-c \\
c' &= c
\end{aligned}
$$

as such $\langle a', b', c' \rangle = \langle a, b, c \rangle$. Then

$$
\begin{aligned}
\partial_2 : C_2 &\to C_1 \\
u &\mapsto a' - 2b' \\
l &\mapsto a'
\end{aligned}
$$

For any 2-chain of the form $pu + ql$ where $p, q \in \mathbb{Z}$, then

$$\partial_2(pu + ql) = (p+q)a' - 2pb'$$

$\partial_2(pu + ql) = 0$ if and only if $p = q = 0$. That is, $\ker \partial_2 = 0$, then

$$H_2(K) = 0$$

Moreover, by setting $p = 0$ and $p = -q$,

$$\operatorname{im} \partial_2 = \langle a', 2b' \rangle$$

Then,

$$H_1(K) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle a', b', c' \rangle}{\langle a', 2b' \rangle} = \mathbb{Z} \oplus \mathbb{Z}/2$$

5.2 Some prerequisites

Proposition 2. *The set of abelian groups, $\oplus$ and $\otimes$ form a commutative semiring with additive identity 0 and multiplicative identity $\mathbb{Z}$*

In the following, let A, B, C be abelian groups, $n \in \mathbb{N}$

- $a := \mathbb{Z}, b := \mathbb{Z}/2$
- $A + B := A \oplus B$
- $nA := A \oplus A \oplus ... \oplus A$ (n times)

- $A^n := A \otimes A \otimes ... \otimes A$ (n times)
- $AB := A \otimes B$
- $AB + C := (AB) + C$ (order of operations)
- $(A+B)C = AC + BC, C(A+B) = CA + CB$ (left/right distributivity)
- $n(AB) = (nA)B$ write nAB
- $A + B = B + A, AB = BA$ (commutativity)
- $aA = Aa = A$
- $b^2 = b$ ($\mathbb{Z}/m \otimes \mathbb{Z}/n = \mathbb{Z}/gcd(m,n)$)
- $T(A,B) := \mathrm{Tor}_1^{\mathbb{Z}}(A,B)$
- $T(a,A) = T(A,a) = 0$ ($a = \mathbb{Z}$ is free over $\mathbb{Z}$)
- $T(b,b) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) = \ker(2 : \mathbb{Z}/2 \to \mathbb{Z}/2) = \mathbb{Z}/2 = b$
- $T(A,B) = T(B,A)$ (Tor symmetric)
- $T(A+B,C) = T(A,C) + T(B,C)$ (Tor commutes with direct product)
- $T(nA,B) = nT(A,B)$

Theorem 2 (Künneth theorem)**.** *If R is an PID, $C_\bullet(X), C_\bullet(Y)$ are degree-wise free then the short exact sequence below is natural and splits*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \longrightarrow H_n(X \times Y;R) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X;R), H_q(Y;R)) \longrightarrow 0$$

Hence,

$$H_n(X \times Y;R) = \left(\bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X;R), H_q(Y;R))\right)$$

Homology of K:

$$\begin{aligned} H_0(K) &= a \\ H_1(K) &= a + b \\ H_q(K) &= 0 && 3 \leq q \end{aligned}$$

5.3 Homology of $K^2 = K \times K$

$$\begin{aligned}
H_0(K^2) &= H_0(K)H_0(K) \\
&= aa = a \\
H_1(K^2) &= H_1(K)H_0(K) + H_0(K)H_1(K) + T(H_0(K), H_0(K)) \\
&= H_1(K) + H_1(K) && (H_0(K) = a) \\
&= 2a + 2b \\
H_2(K^2) &= H_1(K)H_1(K) + T(H_1(K), H_0(K)) + T(H_0(K), H_1(K)) \\
&= H_1(K)H_1(K) && (H_0(K) = a) \\
&= (a+b)(a+b) \\
&= a^2 + ba + ab + b^2 \\
&= a + 2b + b^2 \\
&= a + 3b \\
H_3(K^2) &= T(H_1(K), H_1(K)) \\
&= T(a+b, a+b) \\
&= T(a,a) + T(a,b) + T(b,a) + T(b,b) \\
&= T(b,b) \\
&= b
\end{aligned}$$

In summary

$$
\begin{aligned}
&H_0(K^2) = a \\
&H_1(K^2) = 2a + 2b \\
&H_2(K^2) = a + 3b \\
&H_3(K^2) = b \\
&H_q(K^2) = 0 && \text{for all } 4 \leq q
\end{aligned}
$$

5.4 Homology of $K^3 = K \times K \times K$

$$
\begin{aligned}
H_0(K^3) &= H_0(K^2) \otimes H_0(K) \\
&= aa = a \\
H_1(K^3) &= H_1(K^2)H_0(K) + H_0(K^2)H_1(K) + T(H_0(K^2), H_0(K)) \\
&= H_1(K^2) + H_1(K) && (H_0(K) = H_0(K^2) = a) \\
&= 2a + 2b + a + b \\
&= 3a + 3b \\
H_2(K^3) &= H_2(K^2)H_0(K) + H_1(K^2)H_1(K) \\
&+ T(H_1(K^2), H_0(K)) + T(H_0(K^2), H_1(K)) \\
&= H_2(K^2) + H_1(K^2)H_1(K) && (H_0(K) = H_0(K^2) = a) \\
&= a + 3b + (2a + 2b)(a + b) \\
&= a + 3b + 2a(a + b) + 2b(a + b) \\
&= a + 3b + 2a^2 + 2ab + 2ba + 2b^2 \\
&= a + 3b + 2a + 2b + 2b + 2b \\
&= 3a + 9b \\
H_3(K^3) &= H_3(K^2)H_0(K) + H_2(K^2)H_1(K) \\
&+ T(H_2(K^2), H_0(K)) + T(H_1(K^2), H_1(K)) + T(H_0(K^2), H_2(K)) \\
&= H_3(K^2) + H_2(K^2)H_1(K) + T(H_1(K^2), H_1(K)) && (H_0(K) = H_0(K^2) = a) \\
&= b + (a + 3b)(a + b) + T(2a + 2b, a + b) \\
&= b + a^2 + ab + 3ba + 3b^2 + 2T(a, a) + 2T(a, b) + 2T(b, a) + 2T(b, b) \\
&= b + a + b + 3b + 3b + b \\
&= a + 9b \\
H_4(K^3) &= H_3(K^2)H_1(K) + T(H_3(K^2), H_0(K)) + T(H_2(K^2), H_1(K)) \\
&= H_3(K^2)H_1(K) + T(H_2(K^2), H_1(K)) && (H_0(K) = a) \\
&= b(a + b) + T(a + 3b, a + b) \\
&= ba + b^2 + T(a, a) + T(a, b) + 3T(b, a) + 3T(b, b) \\
&= b + b + 3b \\
&= 5b \\
H_5(K^3) &= T(H_3(K^2), H_1(K)) \\
&= T(b, a + b) \\
&= T(b, a) + T(b, b) \\
&= b
\end{aligned}
$$

In summary,

$$
\begin{aligned}
&H_0(K^3) = a = \mathbb{Z} \\
&H_1(K^3) = 3a + 3b = 3\mathbb{Z} \oplus 3(\mathbb{Z}/2) \\
&H_2(K^3) = 3a + 9b = 3\mathbb{Z} \oplus 9(\mathbb{Z}/2) \\
&H_3(K^3) = a + 9b = \mathbb{Z} \oplus 9(\mathbb{Z}/2) \\
&H_4(K^3) = 5b = 5(\mathbb{Z}/2) \\
&H_5(K^3) = b = \mathbb{Z}/2 \\
&H_q(K^3) = 0 && \text{for all } 6 \leq q
\end{aligned}
$$