ma5209 - algebraic topology

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Chapter 1

CATEGORY AND HOMOLOGICAL ALGEBRA

1.1 BASIC HOMOLOGICAL ALGEBRA

Definition 1 (chain complex). *A chain complex $A_\bullet$ is a sequence of abelian groups and homomorphisms, namely boundary operator*

$$... \xleftarrow{\partial} A_{-2} \xleftarrow{\partial} A_{-1} \xleftarrow{\partial} A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} A_2 \xleftarrow{\partial} ...$$

such that $\partial^2 = 0$, that is, $\mathrm{im}(\partial : A_{n+1} \to A_n) \subseteq \ker(\partial : A_n \to A_{n-1})$

Definition 2 (cycle, boundary, homology group). *Given a chain complex $A_\bullet$ with boundary operator ∂.*

- *An n-cycle is an element of $Z_n(A)$ defined by*

$$Z_n(A_\bullet) = \ker(\partial : A_n \to A_{n-1})$$

- *An n-boundary is an element of $B_n(A)$ defined by*

$$B_n(A_\bullet) = \mathrm{im}(\partial : A_{n+1} \to A_n)$$

- *The nth homology group of A is the quotient*

$$H_n(A_\bullet) = \frac{Z_n(A_\bullet)}{B_n(A_\bullet)}$$

Definition 3 (chain map). *Let $A_\bullet, B_\bullet$ be chain complexes. A chain map $g : A_\bullet \to B_\bullet$ is a sequence of homomorphisms such that $\partial g = g\partial$. The following square commutes*

$$\begin{tikzcd} A_n \arrow[r, "\partial"] \arrow[d, "g"] & A_{n-1} \arrow[d, "g"] \\ B_n \arrow[r, "\partial"] & B_{n-1} \end{tikzcd}$$

Proposition 1. *Let $g : A \to B$ be a chain map. Then, g determines a homomorphism $H_n(g) = g_* : H_n(A) \to H_n(B)$ for all n*

$$\begin{tikzcd} A_n \arrow[r, "g"] \arrow[d, "\subseteq"] & B_n \arrow[d, "\subseteq"] \\ Z_n(A) \arrow[r, "g"] \arrow[d, two heads] & Z_n(B) \arrow[d, two heads] \\ H_n(A) \arrow[r, "g_*"] & H_n(B) \end{tikzcd}$$

Proof.
($g : Z_n(A) \to Z_n(B)$ well-defined - g maps cycles to cycles) Let $\sigma \in Z_n(A)$, then $\partial\sigma = 0$ and $\partial g\sigma = g\partial\sigma = g0 = 0$
($g : B_n(A) \to B_n(B)$ well-defined - g maps boundaries to boundaries) Let $\sigma \in B_n(A)$, then $\sigma = \partial b$ where $b \in C_{n+1}(A)$, and $g\sigma = g\partial b = \partial gb \in B_n(B)$

($g_* : H_n(A) \to H_n(B)$ well-defined) Any element of $Z_n(A)$ in equivalence class $[a] \in H_n(A)$ can be written as $a + \partial c$, then $g(a+\partial c) = ga + g\partial c = ga$. That is, g maps elements from the same equivalence class to the same value.

($g_* : H_n(A) \to H_n(B)$ is a homomorphism) Let $a + \partial c, b + \partial d \in Z_n(A)$ be in the equivalence classes $[a], [b] \in H_n$. Then $g(a + \partial c + b + \partial d) = g(a + \partial c) + g(b + \partial d)$. Therefore, $g_*([a] + [b]) = g_*[a] + g_*[b]$ □

Proposition 2. *Let $1 : A \to A, f : B \to C, g : A \to B$ be chain maps.*

1. $H_n(1) = 1$
2. $H_n(fg) = H_n(f)H_n(g)$

Proof.
(1) obvious
(2) $H_n(fg)[a] = [fga] = H_n(f)[ga] = H_n(f)H_n(g)[a]$ □

Definition 4 (chain homotopy)**.** *Let $g_0, g_1 : A_\bullet \to B_\bullet$ be chain maps. A chain homotopy from g_0 to g_1 is a sequence of homomorphisms $h : A_n \to B_{n+1}$ such that $\partial h + h\partial = g_0 - g_1$*

$$\begin{tikzcd}
\dots & A_0 \arrow[l, "\partial"'] \arrow[dr, "h"] & A_1 \arrow[l, "\partial"'] \arrow[dr, "h"] & A_2 \arrow[l, "\partial"'] \arrow[dr, "h"] & \dots \arrow[l, "\partial"'] \\
\dots & B_0 \arrow[l, "\partial"'] & B_1 \arrow[l, "\partial"'] & B_2 \arrow[l, "\partial"'] & \dots \arrow[l, "\partial"']
\end{tikzcd}$$

If there is a chain homotopy from g_0 to g_1, we said g_0 and g_1 are chain homotopic or of the same chain homotopy type.

Proposition 3. *Two chain homotopic maps induce the same map in homology. That is, if g_0, g_1 are chain homotopic, then $g_{0*} = g_{1*}$*

Proof. Let $a \in Z_n(A)$, then $\partial a = 0$ and $g_0(a) = g_1(a) + \partial h a + h\partial a = g_1(a) + \partial h a$. So, $[g_0(a)] = [g_1(a)]$ □

Definition 5 (chain homotopy equivalence)**.** *A map $f : X \to Y$ is a chain homotopy equivalence if there exists $g : Y \to X$ such that $fg : Y \to Y$ is chain homotopic to 1_Y and $gf : X \to X$ is chain homotopic to 1_X*

Definition 6 (short exact sequence)**.** *A sequence of groups and homomorphisms*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence if one of the following equivalent conditions satisfies

- *(change needed) p is surjective and $i =$ inclusion of $\ker p$*
- *(change needed) i is injective and $p =$ projection onto $\operatorname{coker} i$*
- *i is injective, p is surjective, and $\operatorname{im} i = \ker p$*

Definition 7 (exact)**.** *Given sequences of groups*

- $A \xrightarrow{i} B \xrightarrow{p} C$ *is exact at B if $\operatorname{im} i = \ker p$*
- $\dots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \dots$ *is exact if it is exact at every group*

Theorem 1. *Short exact sequence of chain complexes induces long exact sequence in homology. That is,*

Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ *be a short exact sequence of chain complexes and i, p be chain maps.*

$$\begin{tikzcd}
& \dots \arrow[d, "\partial"] & \dots \arrow[d, "\partial"] & \dots \arrow[d, "\partial"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "p"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "p"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n-1} \arrow[r, "p"] \arrow[d, "\partial"] & C_{n-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
& \dots & \dots & \dots &
\end{tikzcd}$$

Then there exists connecting homomorphisms ∂ such that the sequence

$$
\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial"'] \\
H_{n+1}(A) \arrow[r, "i_*"] & H_{n+1}(B) \arrow[r, "p_*"] & H_{n+1}(C) \arrow[dll, "\partial"'] \\
H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "p_*"] & H_n(C) \arrow[dll, "\partial"'] \\
H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "p_*"] & H_{n-1}(C) \arrow[dll, "\partial"'] \\
\cdots & &
\end{tikzcd}
$$

is exact.

Proof.

(1 defining of $\partial : H_{n+1}(C) \to H_n(A)$)

$$
\begin{tikzcd}
n+1 & & b \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & c \arrow[d, "\partial"] \\
n: & a \arrow[r, mapsto, "i"] \arrow[d, "\partial"] & \partial b \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & 0 \\
n-1: & \partial a \arrow[r, mapsto, "i"] & \partial^2 b = 0 &
\end{tikzcd}
$$

Given $[c] \in H_{n+1}(C)$, (1) take any representative $c \in Z_{n+1}(C)$. As $p : B_{n+1} \to C_{n+1}$ is surjective, (2) take any $b \in B_{n+1}$ such that $pb = c$. As $p\partial b = \partial pb = \partial c = 0$ and $\ker(p : B_n \to C_n) = \operatorname{im}(i : A_n \to B_n)$, take $a \in A_n$ such that $ia = \partial b$, this choice is unique as i is injective. $i\partial a = \partial ia = \partial^2 b = 0$, as i is an injective homomorphism, $\partial a = 0$, then $a \in Z_n(A)$. The construction is done by $[c] \mapsto [a]$

We will verify that the homology class of a does not depend on (1) the representative c and (2) the value of b.

(1.1 $[a]$ does not depend on the choice of b)

$$
\begin{tikzcd}
n+1: & d \arrow[r, mapsto, "i"] \arrow[d, "\partial"] & b - b_1 \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & 0 \\
n: & \partial d = a - a_1 \arrow[r, "i"] & \partial b - \partial b_1 &
\end{tikzcd}
$$

Let $b_1 \in B_{n+1}$ such that $pb_1 = c$ and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that a and a_1 belong to the same homology class, that is $a - a_1$ is a boundary. As $p(b - b_1) = c - c = 0$, there exists a unique $d \in A_{n+1}$ such that $id = b - b_1$. Then, $i\partial d = \partial id = \partial(b - b_1) = \partial b - \partial b_1 = ia - ia_1 = i(a - a_1)$. As i is injective, $a - a_1 = \partial d$, that is, $a - a_1$ is a boundary.

(1.1 $[a]$ does not depend on the choice of c)

$$
\begin{tikzcd}
n+2: & & & & f \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & e \arrow[d, "\partial"] \\
n+1: & d \arrow[r, mapsto, "i"] \arrow[d, "\partial"] & b - b_1 - \partial f \arrow[r, mapsto, "p"] & 0 & \partial f \arrow[r, "p"] & c - c_1 = \partial e \\
n: & a - a_1 = \partial d & & & &
\end{tikzcd}
$$

Let $c_1 = c - \partial e$ for $e \in C_{n+2}$ be in the same homology class with c, $b_1 \in B_{n+1}$ such that $pb_1 = c_1$, and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that a and a_1 belong to the same homology class, that is $a - a_1$ is a boundary. Let $f \in B_{n+2}$ such that $pf = e$. Then, $p\partial f = \partial pf = \partial e$. Hence, $p(b - b_1 - \partial f) = c - c_1 - \partial e = 0$. There exists unique $d \in A_{n+1}$ such that $id = b - b_1 - \partial f$, then $i\partial d = \partial id = \partial(b - b_1 - \partial f) = \partial b - \partial b_1 - \partial^2 f = ia - ia_1 = i(a - a_1)$. As i is injective, $a - a_1 = \partial d$, that is, $a - a_1$ is a boundary.

(1.3 ∂ is a homomorphism)

Let $c_1 \in Z_{n+1}(C)$, $b_1 \in B_{n+1}$ such that $pb_1 = c_1$, and $a_1 \in A_n$ such that $ia_1 = \partial b_1$. We want to prove that $\partial(c + c_1) = a + a_1$. Indeed, this is done verifying $p(b + b_1) = c + c_1$ and $i(a + a_1) = \partial(b + b_1)$.

(2 exactness at $H_n(A), H_n(B), H_n(C)$) It is equivalent to $\operatorname{im}\partial = \ker i_*, \operatorname{im} i_* = \ker p_*, \operatorname{im} p_* = \ker \partial$

(2.1 $\operatorname{im}\partial \subseteq \ker i_* \iff$ for all $[c] \in H_{n+1}(C)$, $i_*\partial[c] = 0$) $i_*\partial[c] = i_*[a] = [ia] = [\partial b] = 0$

(2.2 $\operatorname{im} i_* \subseteq \ker p_* \iff$ for all $[a] \in H_n(A)$, $p_* i_*[a] = 0$) $p_* i_*[a] = p_*[ia] = [pia] = 0$

(2.3 $\operatorname{im} p_* \subseteq \ker \partial \iff$ for all $[b] \in H_n(B)$, $\partial p_*[b] = 0$) $\partial p_*[b] = \partial[pb] = i^{-1}(\partial^* b)$ where $\partial^* : B_n \to B_{n-1}$ denotes the boundary operator. As $b \in Z_n(B)$, $\partial^* b = 0$ and i is an injective homomorphism, then $\partial p_*[b] = i^{-1}(\partial^* b) = i^{-1}(0) = 0$

(2.4 $\operatorname{im}\partial \supseteq \ker i_* \iff$ for all $[\overline{a}] \in H_n(A)$, if $i_*[\overline{a}] = 0$, then there exists $[c] \in H_{n+1}(C)$ such that $\partial[c] = [\overline{a}]$)
$i_*[\overline{a}] = 0 \implies [i\overline{a}] = 0$, that is $i\overline{a}$ is a boundary, then there exists $b \in B_n$ such that $\partial^* b = i\overline{a}$. Take $c = pb$.
(2.5 $\operatorname{im} i_* \supseteq \ker p_*$) TODO - look at the notes from prof
(2.6 $\operatorname{im} p_* \supseteq \ker \partial \iff$ for all $[c] \in H_n(C)$, if $\partial[c] = 0$, then there exists $[\overline{b}] \in H_n(B)$ such that $p_*[\overline{b}] = [c]$)

$$\begin{tikzcd}
n: & d \arrow[d, "\partial"] & b \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & c & d \arrow[r, mapsto, "i"] \arrow[d, "\partial"] & id \arrow[r, mapsto, "p"] \arrow[d, "\partial"] & 0 \\
n-1: & a \arrow[r, mapsto, "i"] & ia = \partial b & & a \arrow[r, mapsto, "i"] & ia = \partial b &
\end{tikzcd}$$

As p is surjective, let $b \in B_n$ such that $pb = c$, $a \in A_{n-1}$ such that $ia = \partial^* b$, then $[a] = \partial[c] = 0$, that is a is a boundary. Then, there exists $d \in A_n$ such that $a = \partial^* d$. Take $\overline{b} = b - id \in B_n$, then $p_*[\overline{b}] = [p(b - id)] = [pb - pid] = [pb] = [c]$, and $\overline{b} \in Z_n(B)$ because $\partial^* \overline{b} = \partial^*(b - id) = \partial^* b - \partial^* id = \partial^* b - i\partial^* d = \partial^* b - ia = 0$

□

Definition 8 (short exact sequence splits)**.** *A short exact sequence splits if TODO*

1.2 BASIC CATEGORY THEORY

1.2.1 CATEGORY, FUNCTOR, NATURAL TRANSFORMATION

Definition 9 (category)**.** *A category C is a class $\operatorname{ob} C$ of objects such that*

- *for all $X, Y \in \operatorname{ob} C$, a class $C(X, Y)$ (or $\hom_C(X, Y), \hom(X, Y)$) of morphisms*
- *for all $X \in \operatorname{ob} C$, $1_X \in C(X, X)$: the identity morphism*
- *for all $X, Y, Z \in \operatorname{ob} C$, there exists a function*

$$\begin{aligned} \circ : C(Y, Z) \times C(X, Y) &\to C(X, Z) \\ (g, f) &\mapsto gf \end{aligned}$$

  $f \in C(X, Y)$ is also written as $f : X \to Y$

and

1. *Let $f : X \to Y$, $1_Y f = f$, $f 1_X = f$*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "1_Y f = f"'] & Y \arrow[d, "1_Y"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "1_X"'] \arrow[dr, "f 1_X = f"] & \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

2. *Let $f : X \to Y, g : Y \to Z, h : Z \to W$, $h(gf) = (hg)f$*

Remark 1. *:*
Some remarks on categories

- Set*: the category of sets and functions*
- Top*: the category of topological spaces and continuous maps*
- Grp*: the category of groups and homomorphisms*
- Ab*: the category of abelian groups and homomorphisms*

Some remarks on morphisms: given a morphism $f : A \to B$

- *f is said to be a monomorphism if for all $g_1, g_2 : X \to A$, $fg_1 = fg_2 \implies g_1 = g_2$*
- *f is said to be an epimorphism if for all $g_1, g_2 : B \to X$, $g_1 f = g_2 f \implies g_1 = g_2$*
- *f is said to be an isomorphism if there exists $g : B \to A$ such that $fg = 1_B$ and $gf = 1_A$*

Definition 10 (functor)**.** *Let C, D be categories. A functor from C to D, $F : C \to D$ is*

- $F : \mathrm{ob}\, C \to \mathrm{ob}\, D$
- *for all* $X, Y \in \mathrm{ob}\, C$*, there is a function* $F : C(X, Y) \to C(F(X), F(Y))$ *such that*
  - *for all* $X \in \mathrm{ob}\, C$*,* $F(1_X) = 1_{F(X)}$
  - *for all* $f : X \to Y, g : Y \to Z$*,* $F(gf) = F(g)F(f)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \xrightarrow{F} \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(gf)=F(g)F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

Remark 2. *Some remarks on functors*

- *identity functor:* $1 : C \to C$
- *forgetful functors:* $U : \mathrm{Grp} \to \mathrm{Set}$*,* $U : \mathrm{Top} \to Set$
- *singular homology:* $H_n : \mathrm{Top} \to \mathrm{Ab}$
- *free abelian group functor:* $\mathbb{Z} : \mathrm{Set} \to \mathrm{Ab}$

Definition 11 (natural transformation)**.** *Let* C, D *be categories and* $F, G : C \to D$ *be functors. A natural transformation from* F *to* G *is an assignment of each* $X \in \mathrm{ob}\, C$ *to a morphism* $\theta_X : F(X) \to G(X)$ *in* D *such that for all* $f : X \to Y$ *in* C*. The diagram below commutes*

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

Notation:

- $\mathrm{ob}\,\mathrm{Fun}(C, D)$*: functors* $C \to D$
- $\mathrm{Fun}(C, D)(F, G)$*: natural transformations* $F \to G$

Remark 3. *Some remarks on natural transformations*

- *The boundary operator* ∂ *is a natural transformation from* C_n *to* C_{n-1} *where* C_n *is the functor from the* Top *to the category of free abelian group of* n*-chains*

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d, "C_n(f)=f_\#"'] & C_{n-1}(X) \arrow[d, "C_{n-1}(f)=f_\#"] \\ C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \end{tikzcd}$$

- *The connecting homomorphism* ∂ *is a natural transformation from* $H_n(C)$ *to* $H_{n-1}(A)$ *where* $H_n(C)$ *and* $H_{n-1}(A)$ *are two functors from the category of short exact sequence of chain complexes to* Ab *and defined by*
  - $H_n(C)$*(upper chain)* $= H_n(C_\bullet)$
  - $H_n(C)$*(lower chain)* $= H_n(C'_\bullet)$
  - $H_n(C)(f) = H_n(f) = f_*$
  - $H_{n-1}(A)$*(upper chain)* $= H_{n-1}(A_\bullet)$
  - $H_{n-1}(A)$*(lower chain)* $= H_{n-1}(A'_\bullet)$
  - $H_{n-1}(A)(f) = H_n(f) = f_*$

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r] \arrow[d, "f"] & B_\bullet \arrow[r] \arrow[d, "f"] & C_\bullet \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r] & B'_\bullet \arrow[r] & C'_\bullet \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} H_n(C_\bullet) \arrow[r, "\partial"] \arrow[d, "H_n(C)(f)=f_*"'] & H_{n-1}(A_\bullet) \arrow[d, "H_{n-1}(A)(f)=f_*"] \\ H_n(C'_\bullet) \arrow[r, "\partial"'] & H_{n-1}(A'_\bullet) \end{tikzcd}$$

1.2.2 UNIVERSAL MAPPING PROPERTY

Definition 12 (product)**.** *Let $X, Y \in \operatorname{ob} C$, the product an object Z together with $p_1 : Z \to X$, $p_2 : Z \to Y$. Moreover, the triplet (Z, p_1, p_2) is universal, that is, for each diagram $X \xleftarrow{f_X} W \xrightarrow{f_Y} Y$, there exists a unique $h : W \to Z$ such that the diagram commutes*

$$\begin{tikzcd}
 & W \arrow[dl, "f_X"'] \arrow[d, dashed, "h"] \arrow[dr, "f_Y"] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

The coproduct is written by $Z = X \times Y$

Definition 13 (coproduct)**.** *Let $X, Y \in \operatorname{ob} C$, the coproduct an object Z together with $i_1 : X \to Z$, $i_2 : Y \to Z$. Moreover, the triplet (Z, i_1, i_2) is universal, that is, for each diagram $X \xrightarrow{f_X} W \xleftarrow{f_Y} Y$, there exists a unique $h : Z \to W$ such that the diagram commutes*

$$\begin{tikzcd}
 & W & \\
X \arrow[ur, "f_X"] \arrow[r, "i_1"'] & X \amalg Y \arrow[u, dashed, "h"'] & Y \arrow[ul, "f_Y"'] \arrow[l, "i_2"]
\end{tikzcd}$$

The coproduct is written by $Z = X \amalg Y$

Remark 4. *Some remarks on coproduct*

- *direct sum on abelian groups is a coproduct*
- *disjoint union on sets is a coproduct*
- *free product on groups is a coproduct*
- *coproduct is unique upto isomorphism*

  Proof.
  Let $(Z, i_1, i_2), (Z', i'_1, i'_2)$ be both coproduct of X, Y. Using universal property, we can construct unique $h : Z \to Z'$ and $h' : Z' \to Z$ such that the diagram below commutes

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "i'_1"] \arrow[ddr, "i_1"'] & Z \arrow[d, dashed, "h"] & Y \arrow[l, "i_2"'] \arrow[dl, "i'_2"'] \arrow[ddl, "i_2"] \\
 & Z' \arrow[d, dashed, "h'"] & \\
 & Z &
\end{tikzcd}$$

  By uniqueness of h, h', we must have $h'h = 1_Z$. Similarly, $hh' = 1_{Z'}$ ∎

- *Construction of coproduct from pushout: coproduct of X, Y is the pushout of $X \longleftarrow 0 \longrightarrow Y$ where 0 is the initial object.*

Definition 14 (equalizer)**.** *Let $f, g : X \to Y$, the equalizer is an object Q and a map $Q \to X$ such that $fq = gq$. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same condition, there exists a unique map $h : Q' \to Q$ such that the diagram below commutes*

$$\begin{tikzcd}
Q \arrow[r, "q"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \\
Q' \arrow[u, dashed, "h"] \arrow[ur, "q'"'] & &
\end{tikzcd}$$

Definition 15 (coequalizer)**.** *Let $f, g : X \to Y$, the coequalizer is an object Q and a map $q : Y \to Q$ such that $qf = qg$. Moreover, the pair (Q, q) is universal, that is, if (Q', q') satisfies the same conditions, there exists a unique map $h : Q \to Q'$ such that the diagram below commutes*

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dashed, "h"] \\ & & Q' \end{tikzcd}$$

Definition 16 (pullback)**.** *The pullback of* $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$ *is* X *and* $X \to B, X \to C$ *such that the diagram below commutes*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & C \arrow[d, "\gamma"] \\ B \arrow[r, "\beta"'] & A \end{tikzcd}$$

Moreover, X *is universal, that is, if there exists* Y *and the commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & C \arrow[d, "\gamma"] \\ B \arrow[r, "\beta"'] & A \end{tikzcd}$$

then there exists a unique map $Y \to X$ *such that the diagram below commutes*

$$\begin{tikzcd} Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \arrow[r] \arrow[d] & C \arrow[d, "\gamma"] \\ & B \arrow[r, "\beta"'] & A \end{tikzcd}$$

Definition 17 (pushout)**.** *The pushout of* $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$ *is* X *and* $A \to X$, $B \to X$ *such that the diagram below commutes*

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

Moreover, X *is universal, that is, if there exists* Y *and the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d] \\ C \arrow[r] & Y \end{tikzcd}$$

then there exists a unique map $X \to Y$ *such that the diagram below commutes*

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

Remark 5. *Some remarks on pushout*

- *Construction of pushout from coproduct and coequalizer: the pushout of* $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$ *, is the coequalizer of*

$$\begin{tikzcd} A \arrow[r, bend left, "i_1\beta"] \arrow[r, bend right, "i_2\gamma"'] & B \amalg C \end{tikzcd}$$

- *Consider the category of sets, then the pushout $X = B \amalg C/\sim$ where the equivalence relation is defined by $\beta(a) \sim \gamma(a)$ for all $a \in A$*
- $\mathbb{R}P^{n-1}$*: real projective space - space of lines through the origin in $\mathbb{R}^n$, $\mathbb{R}P^n$ is the pushout of* $\mathbb{R}P^{n-1} \twoheadleftarrow S^{n-1} \hookrightarrow e^n$ *, the diagram below commutes*

$$\begin{tikzcd}
S^{n-1} \arrow[r, hook] \arrow[d, two heads] & D^n \arrow[d] \\
\mathbb{R}P^{n-1} \arrow[r] & \mathbb{R}P^n
\end{tikzcd}$$

  where S^{n-1} denotes the equator, D^n denotes the closed upper hemisphere.
- $\mathbb{C}P^{n-1}$*: complex projective space - space of lines through the origin in $\mathbb{C}^n$, similarly, $\mathbb{C}P^n$ is the pushout of the diagram below*

$$\begin{tikzcd}
S^{2n-1} \arrow[r, hook] \arrow[d, two heads] & D^{2n} \arrow[d] \\
\mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^n
\end{tikzcd}$$

Definition 18 (base change, cobase change)**.** *:*

1. *if X is the pullback of $C \xrightarrow{\gamma} A \xleftarrow{\beta} B$ then the transform from $B \to A$ to $X \to C$ is called base change.*
2. *if X is the pushout of $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$, then the transform from $A \to B$ to $C \to X$ is called cobase change.*

Definition 19 (direct limit)**.** *Given a category C with*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \ldots$$

the direct limit of this sequence is an object X with maps $g_n : X_n \to X$ for all n such that the diagram below commutes for all n

$$\begin{tikzcd}
 & X_n \arrow[d, "g_n"] \\
X_{n-1} \arrow[ur, "f_n"] \arrow[r, "g_{n-1}"'] & X
\end{tikzcd}$$

Moreover, $(X, g_\bullet)$ is universal, that is, if $(Y, h_\bullet)$ is also a direct limit, then there exists a unique map $h : X \to Y$ such that the diagram below commutes for all n

$$\begin{tikzcd}
 & X_n \arrow[d, "g_n"] \arrow[ddr, "h_n"] & \\
X_{n-1} \arrow[ur, "f_n"] \arrow[r, "g_{n-1}"] \arrow[drr, "h_{n-1}"'] & X \arrow[dr, dashed, "h"] & \\
 & & Y
\end{tikzcd}$$

The direct limit is written by $X = \varinjlim X_n$

Remark 6. *Some remarks on direct limit*

- *Construction of direct limit from pushout: the direct limit of* $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \ldots$ *is X such that X is the pushout of* $X_{n-1} \xleftarrow{1} X_{n-1} \xrightarrow{f_n} X_n$ *for all n*
- *Consider the category of sets, then the direct limit of (X_n) is $\coprod X_n/\sim$ where the equivalence relation is defined by $(x \in X_{n-1}) \sim (f_n(x) \in X_n)$*

$$\begin{tikzcd}
X_n \arrow[r, hook, "i_n"] \arrow[dr, "g_n"] \arrow[ddr, "h_n"'] & \coprod X_n \arrow[d, two heads] \\
 & \coprod X_n/\sim \arrow[d, dashed, "h"] \\
 & Y
\end{tikzcd}$$

- *if each f_n is mono,* $\varinjlim X_n = \bigcup X_n$
- *if each f_n is inclusion of a subspace, then* $\varinjlim X_n$ *describe a topology on* $\bigcup X_n$*, called weak topology, that is,* $U \subseteq \bigcup X_n$ *is said to be open if* $U \cap X_n$ *open for all* n

1.2.3 PREADDITIVE CATEGORY, ADDITIVE FUNCTOR, PROJECTIVE CLASS, DERIVED FUNCTOR

Definition 20 (initial object, terminal object, zero, pointed category, zero map)**.** *Let C be a category. 0 is initial if $\#C(0,X) = 1$ for all X, $*$ is terminal if $\#C(X,*) = 1$ for all X. C is called pointed if it has initial and terminal objects and the unique map $0 \to *$ is an isomorphism. We write 0 for both initial and terminal objects. Consequently, given any two objects M, N there exists a zero map*

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "0"'] & 0 \arrow[r] & N \end{tikzcd}$$

Definition 21 (kernel)**.** *Given a pointed category C, let $f : M \to N$, a kernel of f is a map $i : K \to M$ such that $fi = 0$. Moreover, i is universal, that is, if $j : L \to M$ such that $fj = 0$, then there exists a unique $k : L \to K$ such that the diagram below commutes*

$$\begin{tikzcd} K \arrow[r, "i"'] \arrow[rr, bend left, "0"] & M \arrow[r, "f"'] & N \\ L \arrow[u, dashed, "k"] \arrow[ur, "j"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

The category C has kernels if every morphism has a kernel.

Remark 7. *Some remarks on kernels*

- *Construction of kernel from equalizer: the kernel of $f : X \to Y$ is the equalizer of $f : X \to Y$ and $0 : X \to Y$*

Definition 22 (cokernel)**.** *Given a pointed category C, let $f : M \to N$, a cokernel of f is a map $i : N \to K$ such that $if = 0$. Moreover, i is universal, that is, if $j : N \to L$ such that $jf = 0$, then there exists a unique $k : K \to L$ such that the diagram below commutes*

$$\begin{tikzcd} M \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & N \arrow[r, "i"'] \arrow[dr, "j"'] & K \arrow[d, dashed, "k"] \\ & & L \end{tikzcd}$$

Definition 23 (preadditive category, Ab-enriched category)**.** *A preadditive category is a category $\mathcal{C}$ together with an abelian group structure on* $\operatorname{Hom}(M,N)$ *for all $M, N \in \operatorname{ob}\mathcal{C}$ such that composition map are bilinear. That is,*

$$fh + gh = (f+g)h \text{ and } hf + hg = h(f+g)$$

Preadditive category is also called Ab*-enriched category, or enriched over a monoidal category.*

Definition 24 (additive functor)**.** *A functor F between preadditive categories is additive if*

$$\operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M), F(N))$$

is a homomorphism between two abelian groups.

Definition 25 (P-element, P-exact, P-epic - `https://arxiv.org/pdf/math/9912157.pdf`)**.** *For any two objects P, A, a map $P \to A$ is called P-element of A. If $\mathcal{P}$ is a collection of objects, then $\mathcal{P}$-element is a P-element for some $P \in \mathcal{P}$. Let C be a (locally small) pointed category. Let $P \in \operatorname{ob} C$, a sequence*

$$A \longrightarrow B \longrightarrow C$$

is said to be P-exact if the composite $A \to B \to C$ is the zero map and

$$C(P,A) \longrightarrow C(P,B) \longrightarrow C(P,C)$$

is an exact sequence of pointed sets (based point is the zero map). If $\mathcal{P}$ is a class of objects in C, a $\mathcal{P}$-exact sequence in one which is P-exact for all $P \in \mathcal{P}$. A map $A \to B$ is called P-epic if it induces a surjection of P-elements.

Definition 26 (projective class - `https://arxiv.org/pdf/math/9912157.pdf`)**.** *A projective class on a pointed category C is $(\mathcal{P}, \mathcal{E})$ where $\mathcal{P}$ is a class of objects (projectives) and $\mathcal{E}$ is a class of morphisms (epimorphisms) such that*

1. *$\mathcal{E}$ is precisely the collection of all $\mathcal{P}$-exact sequences*
2. *$\mathcal{P}$ is precisely the collection of all object P such that each sequence in $\mathcal{E}$ is P-exact.*
3. *any map $A \to B$ can be extended to a sequence $P \to A \to A$ in $\mathcal{E}$ with $P \in \mathcal{P}$*

Definition 27 (projective class)**.** *Let C be a pointed category with kernels. A projective class in C is $(\mathcal{P}, \mathcal{E})$ where $\mathcal{P}$ is a class of objects (projectives) and $\mathcal{E}$ is a class of morphisms (epimorphisms) such that*

1. *$P \in \operatorname{ob} C$ lies in $\mathcal{P}$ if and only if for all $f : M \to N$ in $\mathcal{E}$, the map $C(P, M) \to C(P, N)$ defined by $g \mapsto fg$ is onto.*

$$\begin{tikzcd} P \arrow[r, dashed, "g"] \arrow[dr, "fg"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

2. *A morphism $f : M \to N$ lies in $\mathcal{E}$ if and only if for all $P \in \mathcal{P}$, the map $C(P, M) \to C(P, N)$ defined by $g \mapsto fg$ is onto.*

$$\begin{tikzcd} P \arrow[r, dashed, "g"] \arrow[dr, "fg"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

3. *For any object $M \in \operatorname{ob} C$, there exists $P \to M$ in $\mathcal{E}$ such that $P \in P$. This condition is also called C has enough projectives*

Definition 28 (left derived functor)**.** *Given a pointed preadditive category $\mathcal{C}$ with kernels and projective class $(\mathcal{P}, \mathcal{E})$ in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathrm{Ab}$ be an additive functor (it maps chain complexes to chain complexes). The left derived functors of F with respect to $\mathcal{P}$ are $L_n F : \mathcal{C} \to \mathrm{Ab}$ for $n \geq 0$ defined by*

$$L_n F(X) = H_n(F(P_\bullet))$$

where $X \leftarrow P_\bullet$ is a $\mathcal{P}$-projective resolution. (analogous to Tor functor)

Remark 8. *Some remarks on left derived functor*

- *Let $C = \text{R-Mod}$, $\mathcal{P}$ be the categorical projectives (= direct summands of free R-modules), $F : \text{R-Mod} \to \mathrm{Ab}$ defined by $A \mapsto A \oplus_R B$ where $B \in \text{R-Mod}$, then*
$$L_n F(A) = \mathrm{Tor}_n^R(A, B)$$
- *$F(X) \leftarrow L_0 F(X)$ might not be an isomorphism. It is an isomorphism if and only if F is right exact*

Definition 29 (retract)**.** *P is a retract of F is there exists a commutative diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[dr, "1"'] & F \arrow[d] \\ & P \end{tikzcd}$$

The map $P \to F$ is called section and the map $F \to P$

Proposition 4. *Class of projectives $\mathcal{P}$ is closed under coproduct and retract*

1.2.4 AXIOMATISE FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Theorem 2 (fundamental theorem of homological algebra)**.** *Let $\mathcal{C}$ be a pointed category with kernels. Let $(\mathcal{P}, \mathcal{E})$ be a projective class in $\mathcal{C}$.*

1. *For any $M \in \operatorname{ob} \mathcal{C}$, there is an exact sequence w.r.t $(\mathcal{P}, \mathcal{E})$ ($\mathcal{P}$-projective resolution)*

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \ldots$$

   where $P_n \in \mathcal{P}$ and exactness $A \to B \to C$ is defined by the composition being the zero map and $A \to B$ is factored through $\ker(B \to C)$ by an epimorphism, that is, there exists an epimorphism $A \to \ker(B \to C)$ such that the diagram below commutes

$$\begin{tikzcd}
A \arrow[r] \arrow[d, dashed] & B \arrow[r] & C \\
\ker(B \to C) \arrow[ur] & &
\end{tikzcd}$$

2. *Any map $f : M \to M'$ lifts to maps of projectives*

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"'] & P_0 \arrow[l] \arrow[d, dashed] & P_1 \arrow[l] \arrow[d, dashed] & \ldots \arrow[l] \\
0 & M' \arrow[l] & P_0' \arrow[l] & P_1' \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

Furthermore, if $\mathcal{C}$ is preadditive, the lifts are unique upto chain homotopy.

Proof. TODO ☐

1.2.5 YONEDA LEMMA

Definition 30 (functor category, evaluation functor)**.** *Let A, B be categories, the functor category $\mathrm{Fun}(A, B)$ is defined by functors $A \to B$ as objects and natural transformations as morphisms. Let $X \in \mathrm{ob}\, A$, define the evaluation functor $\mathrm{Ev}_X : \mathrm{Fun}(A, B) \to B$ by*

$$\mathrm{Ev}_X(F) = F(X)$$

where $F \in \mathrm{ob}\, \mathrm{Fun}(A, B)$. define application of morphism

Definition 31 (hom functor)**.** *Let C be a locally small category, for any $A \in \mathrm{ob}\, C$, the hom functor $h_A : C \to \mathrm{Set}$ is defined by*

$$\begin{tikzcd}
X \arrow[r, "h_A"] \arrow[d, "f"'] & C(A, X) \arrow[d, "h_A(f)"] \\
Y \arrow[r, "h_A"] & C(A, Y)
\end{tikzcd}$$

where $h_A(f)(g) = fg$ for any $g \in C(A, X)$. h_A is also written as $C(A, -)$

Remark 9. *Some remarks on hom functor*

- *The singular chain complex functor $C_n : \mathrm{Top} \to \mathrm{Ab}$ can be defined by*

$$C_n = \mathbb{Z}\,\mathrm{Top}(\Delta^n, -)$$

Theorem 3 (Yoneda lemma)**.** *Given a locally small category C and $A \in \mathrm{ob}\, C$. Let $F : C \to \mathrm{Set}$ be a functor, then both h_A and F are objects in $\mathrm{Fun}(C, \mathrm{Set})$ and the set of morphisms $h_A \to F$ in $\mathrm{Fun}(C, \mathrm{Set})$ is isomorphic to $F(A)$,*

$$\mathrm{Nat}(h_A, F) \cong F(A)$$

where $\mathrm{Nat}(h_A, F)$ denotes the set of natural transformation from h_A to F. Since, $F(A) = \mathrm{Ev}_A(F)$, then

$$\Phi : \mathrm{Hom}(A, -) \to \mathrm{Ev}_A$$

is a natural isomorphism. We say that Ev_A is corepresentable. (Φ, A) is called corepresentation of Ev_A

Proof.
Let $\Phi : h_A \to F$ be a natural transformation

$$\begin{tikzcd}
A \arrow[ddd, "f"'] & C(A, A) \arrow[rrr, Rightarrow, "\Phi_A"] \arrow[ddd, "h_A(f)"'] & & & F(A) \arrow[ddd, "F(f)"] \\
& & 1 \arrow[r, mapsto] \arrow[d] & u \arrow[d] & \\
& & f \arrow[r, mapsto, dashed] & F(f)(u) & \\
X & C(A, X) \arrow[rrr, Rightarrow, "\Phi_X"] & & & F(X)
\end{tikzcd}$$

By commutativity, $\Phi_X : C(A, X) \to F(X)$ is defined by $f \mapsto F(f)(u)$. That is,

1. for each $u \in F(A)$, there is a unique natural transformation Φ defined by

$$\Phi_X(f) = F(f)(u)$$

2. for each natural transformation Φ, there is unique $u \in F(A)$ defined by

$$u = \Phi_A(1)$$

$\square$

Remark 10. *Some remarks on Yoneda lemma*

- *Take C be a category such objects are sets, take F to be the forgetful functor and $A \in \operatorname{ob} C$. Then, the set of natural transformations $h_A \to F$ is exactly A*
- *in $C = \text{Set}$, each functor $F : C \to \text{Set}$ corresponds to a natural transformation $1 \to F$. From Yoneda lemma, for each natural transformation $1 \to F$, the map $X \to F(X)$ is factored through $X \to C(X, X)$ defined by $x \mapsto 1$*

Lemma 1. *Let C be any category, then $\text{Fun}(C, \text{Ab})$ is preadditive. That is, given any two functors $G, F : C \to \text{Ab}$, the class of natural transformations of $G \to F$ is an abelian group.*

Proof. TODO $\square$

Proposition 5 (Yoneda lemma for Ab-enriched functor)**.** *Given C is preadditive, for any $A \in \operatorname{ob} C$ and functor $F : C \to \text{Ab}$, then we have an isomorphism between abelian groups*

$$\text{Nat}(h_A, F) \cong F(A)$$

Corollary 1. *Given $X, Y \in \operatorname{ob} C$, $X \cong Y$ if and only if $C(X, -) \cong C(Y, -)$*

1.2.6 ACYCLIC MODEL

Remark 11 (kernel of natural transformation between Ab-enriched functors)**.** *Given $\text{Fun}(C, \text{Ab})$ is pointed, has kernels, and preadditive. Let $F, G \in \text{Fun}(C, \text{Ab})$, and $v : G \to F$ be a natural transformation, then*

$$\ker(v : G \to F)(X) = \ker(v_X : G(X) \to F(X))$$

where $X \in \operatorname{ob} C$

Definition 32 (acyclic models)**.** *Let $\mathcal{M}$ be any set of objects in C ("models"). It defines a projective class in $\text{Fun}(C, \text{Ab})$ as follows: an object is projective (relative to $\mathcal{M}$) if it is a retract of a coproduct of $\mathbb{Z}C(M, -)$s. A morphism $Y : G \to F$ is an epimorphism (relative to $\mathcal{M}$) if $G(M) \twoheadrightarrow F(M)$ is surjective for all $M \in \mathcal{M}$*

Remark 12. *Some remarks on acyclic model*

- *$\mathbb{Z}C(M, -)$ is analogous to free $\mathbb{Z}$-module (free abelian group), then the coproduct is direct product and retract is direct summand (A is a summand of B if $B = A \oplus C$)*
- *$C = \text{Top}$, $\mathcal{M} = \{\Delta^n : n \geq 0\}$, this models $\mathcal{M}$ recovers projectives being singular n-chains and left derive functor of evaluation map at X of H_0 is H_n (lecture 16)*

Proof. We will show the that the construction is indeed a projective class in $\text{Fun}(C, \text{Ab})$ by the following

1. If H is a projective and $G \to F$ epimorphism, then there is a lift from $H \to F$ to $H \to G$ (forward direction of (1) and (2))

   It suffices to show for the case if $H = C(M, -)$.

$$\begin{tikzcd} & G \arrow[d] & G(M) \arrow[d, two heads] \\ C(M,-) \arrow[ur, dashed] \arrow[r] & F & F(M) \end{tikzcd}$$

By Yoneda lemma, let natural transformation $C(M,-) \to F$ determined by $v \in F(M)$

$$\begin{aligned} C(M,N) &\to F(N) \\ f &\mapsto F(f)(v) \end{aligned}$$

Since $G(M) \to F(M)$ is surjective, let $u \in G(M)$ such that $u \mapsto v$ by $G(M) \to F(M)$. Construct the natural transformation $C(M,-) \to G$ by

$$\begin{aligned} C(M,N) &\to G(N) \\ f &\mapsto G(f)(u) \end{aligned}$$

The lift make diagram commutes is due to naturality of $G \to F$ as in the diagram below

$$\begin{tikzcd}
 & & G(N) \arrow[dd, two heads] \\
 f \arrow[r, dashed, maps to] \arrow[rd, maps to] & G(f)(u) \arrow[d, maps to] & \\
C(M,N) \arrow[rruu, dashed, "G(-)(u)"] \arrow[rr, "F(-)(v)"'] & F(f)(v) & F(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
G(M) \arrow[rrr, "G(f)"] \arrow[ddd, two heads] & & & G(N) \arrow[ddd, two heads] \\
 & u \arrow[r, maps to] \arrow[d, maps to] & G(f)(u) \arrow[d, dashed, maps to] & \\
 & v \arrow[r, maps to] & F(f)(v) & \\
F(M) \arrow[rrr, "F(f)"'] & & & F(N)
\end{tikzcd}$$

2. If for every projective H there is a lift from $H \to F$ to $H \to G$, then $G \to F$ epimorphism (backward direction of (2))
   Let $H = C(M,-)$,

$$\begin{tikzcd}
 & G \arrow[d] \\
C(M,-) \arrow[r] \arrow[ru, dashed] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
G(M) \arrow[d, two heads] \\
F(M)
\end{tikzcd}$$

FOR LATER - TOO LONG

∎

1.3 EILENBERG-STEENROD-MILNOR AXIOMS

Definition 33 (category of pairs of topological spaces)**.** *The category of pairs of topological spaces* Top_2 *is defined as follows*

- *objects: pair* (X,A) *where* $A \subseteq X$
- *morphisms: continuous map* $f : (X,A) \to (Y,B)$ *that is a continuous map* $f : X \to Y$ *such that* $f(A) \subseteq B$
- *composition law for morphisms: compose a set maps*

Axiom 1 (Eilenberg-Steenrod-Milnor axioms)**.** *Let* C *be a category of pairs of topological spaces* (X,A) *such that if* $(X,A) \in \mathrm{ob}\, C$*, then* $(A,\varnothing) \to (X,\varnothing) \to (X,A)$ *are also in* C*. A homology theory on* C *is a functor* $H_n : \mathrm{ob}\, C \to \mathrm{Ab}, n \in \mathbb{Z}$ *and a natural transformation (connecting homomorphism)* $\partial : H_n(X,A) \to H_{n-1}(A,\varnothing)$ *such that*

1. ***homotopy axiom****: if* $f \simeq g$*, then* $H_n(f) = H_n(g)$ *where* $f,g : (X,A) \to (Y,B)$ *are homotopic if they are homotopic as both a function* $X \to Y$ *and* $A \to B$*.*
2. ***exactness axiom****: for all* $(X,A) \in \mathrm{ob}\, C$*, the sequence below is exact*

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+1}(X,A) \arrow[dll] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll] \\
H_{n-1}(A) \arrow[r] & \dots &
\end{tikzcd}$$

where $H_n(X)$ denotes $H_n(X, \varnothing)$

3. ***excision axiom**: if $(X, A) \in \operatorname{ob} C$ and $U \subseteq \overline{U} \subseteq \operatorname{int} A$ and $(X - U, A - U) \hookrightarrow (X, A)$ in C, then $H_n(X - U, A - U) \cong H_n(X, A)$*

4. ***additivity axiom** (Milnor): homology commutes with coproduct. That is, let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a set of topological spaces. If $(X_\alpha, \varnothing) \in \operatorname{ob} C$ for all $\alpha \in \mathcal{A}$ and $(\coprod X_\alpha, \varnothing) \in \operatorname{ob} C$, then*

$$\begin{tikzcd}
\bigoplus H_n(X_\alpha, \varnothing) \arrow[r, "\cong"] & H_n(\coprod X_\alpha, \varnothing) \\
H_n(X_\alpha, \varnothing) \arrow[u, "i_\alpha"] \arrow[ur, "H_n(i_\alpha)"'] &
\end{tikzcd}$$

5. ***dimension axiom**: $H_n(*, \varnothing) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$*

Remark 13. *Some remarks on Eilenberg-Steenrod axioms*

- *we omit dimension axiom: extraordinary cohomology theory.*
- *varying of dimension axiom: homology with coefficients. Let A be any abelian group, $H_n(*, \varnothing; A) = \begin{cases} A, & n = 0 \\ 0, & n \neq 0 \end{cases}$*

Chapter 2

ALGEBRAIC TOPOLOGY

2.1 SINGULAR HOMOLOGY

2.1.1 DEFINITION OF SINGULAR HOMOLOGY

Definition 34 (n-simplex, barycentric coordinate, barycentre)**.** *The standard n-simplex is the convex hull of $e_0, e_1, ..., e_n$ in $\mathbb{R}^{n+1}$ where $e_0, e_1, ..., e_n$ are the standard basis vectors, denoted by Δ^n,*

$$\Delta^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

We also write $\Delta^n = [e_0, ..., e_n] = [0, ..., n]$. $\{x_0, ..., x_n\}$ is called the barycentric coordinate. The point $b = \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right)$ is called the barycentre.

Definition 35 (singular n-simplex)**.** *A singular n-simplex in X is a continuous function $\sigma : \Delta^n \to X$. The set of singular n-simplicies in X is denoted by $S_n(X)$*

Definition 36 (singular n-chain)**.** *A singular n-chain is an element of the free abelian group $C_n(X) = \mathbb{Z}[S_n(X)]$ generated by S_n. Alternatively, n-chain is a finite formal linear combination of singular n-simplicies with integer coefficients. If $n \leq 0$, $C_n(X) = 0$ is defined to be the trivial group.*

Definition 37 (face map)**.** *The face map $d^i : \Delta^{n-1} \to \Delta^n$ is the affine map sending vertices to vertices, in order, missing i. That is,*

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

Proposition 6. *If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$. Equivalently, $d_i d_j = d_{j-1} d_i$*

Definition 38 (boundary operator)**.** *Define the boundary operator $\partial : C_n(X) \to C_{n-1}(X)$ as a linear extension of $\partial : S_n(X) \to C_{n-1}(X)$*

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ d^i = \sum_{i=0}^{n} (-1)^i d_i \sigma \in C_{n-1}(X)$$

where $\sigma \in S_n(X)$.

Proposition 7. $\partial^2 = 0$

$$\begin{tikzcd} C_{n-1}(X) & C_n(X) \arrow[l, "\partial"'] & C_{n+1}(X) \arrow[l, "\partial"'] \arrow[ll, bend left, "0"] \end{tikzcd}$$

Proposition 8 (singular homology)**.** *The singular n-chains $C_n(X)$ together with the boundary operator ∂ is a chain complex, written as $C_\bullet(X)$. The corresponding n-homology group is denoted by $H_n(X)$.*

2.1.2 PATH-COMPONENT DECOMPOSITION

Define $x \sim y$ if and only if there exists a path in X from x to y, then $\sim$ is an equivalent relation. Each equivalence class of $\sim$ is called a path component of X. The set of path components of X is denoted by $\pi_0(X)$. Since each singular n-simplex $\sigma : \Delta^n \to X$ land entirely in a single path component, we have

$$S_n(X) = \coprod_{\alpha \in \pi_0(X)} S_n(X_\alpha)$$

where X_α denotes the subspace induced by α, and

$$C_n(X) = \bigoplus_{\alpha \in \pi_0(X)} C_n(X_\alpha)$$

Proposition 9. $H_n(X) \cong \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha)$

2.1.3 SOME SPECIAL CASES: $H_0, H_n(*)$

Assume X is path connected, then $Z_0(X) = C_0(X)$, and $H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{C_0(X)}{\operatorname{im}(\partial : C_1(X) \to C_0(X))} = \operatorname{coker} \partial$

Definition 39 (augmentation map)**.** *The augmentation map $\epsilon : C_n(X) \to \mathbb{Z}$ is defined by*

$$\epsilon\left(\sum_{x \in S_n(X)} a_x x\right) = \sum_{x \in S_n(X)} a_x$$

Lemma 2. *Let $\epsilon : C_0(X) \to \mathbb{Z}$, then $\epsilon\partial = 0$*

Proof. For each $\sigma \in S_1(X)$, then $\epsilon\partial\sigma = \epsilon(d_0\sigma - d_1\sigma) = \epsilon d_0\sigma - \epsilon d_1\sigma = 1 - 1 = 0$. ☐

$$\begin{tikzcd} C_1(X) \arrow[r, "\partial"] & C_0(X) \arrow[r, "\epsilon"] & \mathbb{Z} \\ & H_0(X) \arrow[u] \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Lemma 3. *If X is path connected, then the augmentation map induces an isomorphism $\epsilon : H_0(X) \to \mathbb{Z}$*

Proof.

(well-defined) Let a and $a + \partial b$ be two elements from the same cosets of $B_0(X)$, then

$$\epsilon[a + \partial b] = \epsilon(a + \partial b) = \epsilon a = \epsilon[a]$$

(homomorphism)

$$\epsilon([a] + [b]) = \epsilon[a + b] = \epsilon(a + b) = \epsilon a + \epsilon b = \epsilon[a] + \epsilon[b]$$

(surjective) Let any $x \in S_0(X), n \in \mathbb{Z}$, then

$$\epsilon[nx] = \epsilon(nx) = n\epsilon x = n$$

(injective) As ϵ is a homomorphism, we just need to prove that the kernel of ϵ is 0, that is, for all $c \in C_0(X)$, if $\epsilon c = 0$, then $[c] = 0$, that is, $c \in B_0(X)$. Let $c = \sum_{x \in S_0(X)} a_x x$ and $\epsilon c = 0$, then $\sum_{x \in S_0(X)} a_x = 0$. We can rewrite

$$c = \sum_{x \in S_0(X)} a_x x = (x_1 + ... + x_n) - (y_1 + ... + y_n)$$

where $x_1, ..., x_n, y_1, ..., y_n \in S_0(X)$. For each $i = 1, ..., n$, pick $\sigma_i \in S_1(X)$ such that $d_0\sigma_i = x_i$ and $d_1\sigma_i = y_i$ (pick $\sigma_i : \Delta^1 \to X$ mapping the end points of Δ^1 to x_i and y_i), then $\sum_{i=1}^n \sigma_i \in C_1(X)$, and

$$\partial\left(\sum_{i=1}^n \sigma_i\right) = \sum_{i=1}^n (d_0\sigma_i - d_1\sigma_i) = \sum_{i=1}^n (x_i - y_i) = c$$

☐

Corollary 2. $H_0(X) = \bigoplus_{a \in \pi_0(X)} \mathbb{Z}$ *for any space X*

Proposition 10. $H_n(*) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$ *where* $*$ *denotes the space of one point.*

Proof. For any $n = 0, 1, ..., S_n(*) = *$ then $C_n(*) = \mathbb{Z}$. For $n > 0$, and $* \in S_n(X)$,

$$\partial * = \sum_{i=0}^{n} (-1)^i d_i * = \sum_{i=0}^{n} (-1)^i * = \begin{cases} 0, & n \text{ odd} \\ *, & n \text{ even} \end{cases}$$

We have the diagram

$$0 \xleftarrow{\partial=0} C_0(*) \xleftarrow{\partial=0} C_1(*) \xleftarrow{\partial=1} C_2(*) \xleftarrow{\partial=0} C_3(*) \xleftarrow{\partial=1} ...$$

$H_0(*) = \mathbb{Z}$ by the previous explanation. If $n > 0$ (n odd or even), $Z_n(*) = B_n(*) = C_n(*) = \mathbb{Z}$, then $H_n(*) = 0$.

□

2.1.4 INDUCED FUNCTION BETWEEN TWO TOPOLOGICAL SPACES

Proposition 11 (naturality of induced map)**.** *A function $f : X \to Y$ induces a function $f_\# : C_n(X) \to C_n(Y)$ defined by a linear extension of $f_\# : S_n(X) \to S_n(Y)$ with $f_\# \sigma = f\sigma$*

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "f_\#\sigma = f\sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Then, $f_\# \partial = \partial f_\#$, that is, the following square commutes

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[d, "f_\#"] \\ C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \end{tikzcd}$$

Proof. Let $\sigma \in S_n(X)$, then

$$f_\# \partial \sigma = f_\# \sum_{i=0}^{n} (-1)^i \sigma d^i = f \sum_{i=0}^{n} (-1)^i \sigma d^i = \sum_{i=0}^{n} (-1)^i f \sigma d^i = \partial f_\# \sigma$$

□

Remark 14 (chain map)**.** *The function $f_\# : C_n(X) \to C_n(Y)$ induced from $f : X \to Y$ is a chain map of from chain complex $C_\bullet(X)$ to chain complex $C_\bullet(Y)$. The induced map in homology is written as $H_n(f) = f_* : H_n(A) \to H_n(B)$*

2.1.5 HOMOTOPY

Definition 40 (homotopy)**.** *Let $f_0, f_1 : X \to Y$ be continuous maps. A homotopy from f_0 to f_1 is a continuous map $F : X \times I \to Y$ where $I = [0, 1]$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Being homotopic is an equivalence relation on maps: $(X, Y) \to [X, Y]$ where (X, Y) denotes the set of all maps from X to Y and $[X, Y]$ denotes the equivalence classes of homotopy of maps from X to Y. f_0 and f_1 being homotopic is denoted by $f_0 \simeq f_1$.*

Theorem 4 (homotopy invariance)**.** *Two homotopic maps induce the same map in homology. That is, if $f_0 \simeq f_1$, then $H_n(f_0) = H_n(f_1) : H_n(X) \to H_n(Y)$ for all n*

Proof. Consequence of Proposition 3 and Proposition 13 □

prism operator
Let $f_0, f_1 : X \to Y$ be homotopic, the diagram below commutes

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f_0"] & \\ X \times I \arrow[r, "F"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "f_1"'] & \end{tikzcd}$$

where $i_0 : x \mapsto (x, 0)$, and $i_1 : x \mapsto (x, 1)$

Definition 41 (prism operator). *Given homotopy $F : X \times I \to Y$, the prism operator $h_F : C_n(X) \to C_{n+1}(Y)$ is defined by a linear extension of $h_F : S_n(X) \to C_{n+1}(Y)$ as follows. Let $\sigma \in S_n(X)$,*

$$\begin{tikzcd}
\Delta^n \times I \arrow[r, "\sigma\times 1"] & X \times I \arrow[r, "F"] & Y \\
\Delta^{n+1} \arrow[u, "P_n = h_{\Delta^n}(1)"] \arrow[ur, "h_X(\sigma)"] \arrow[urr, "h_F(\sigma)"'] & &
\end{tikzcd}$$

Let $[v_0, ..., v_n], [w_1, ..., w_n]$ be the bases of the prism $\Delta^n \times I$. Define $\sigma_i : \Delta^{n+1} \to \Delta^n \times I$ be the affine map from Δ^{n+1} to the simplex $[v_0, v_1, ..., v_i, w_i, w_{i+1}, ..., w_n]$. Define $P_n = \sum_{i=0}^{n} (-1)^i \sigma_i$, and $h_F(\sigma) = F(\sigma \times 1)P_n$ is then defined accordingly.

Proposition 12. *$\partial P_n = -P_{n-1}\partial + i_1 - i_0$ and $\partial h_F = -h_F\partial + f_1 - f_0$ where $i_1 : \Delta^n \mapsto (\Delta^n, 1), i_0 : \Delta^n \mapsto (\Delta^n, 0)$*

Proof. TODO □

Proposition 13. *Two homotopic maps induce chain homotopic maps. That is, if $f_0 \simeq f_1 : X \to Y$, then $f_{0\#}, f_{1\#} : C_n(X) \to C_n(Y)$ are chain homotopic*

$$\begin{tikzcd}
C_{n-1}(X) \arrow[dr, "h_F"] & C_n(X) \arrow[l, "\partial"'] \arrow[d, "f_{0\#}-f_{1\#}"] \arrow[dr, "h_F"] & \\
& C_n(Y) & C_{n+1}(Y) \arrow[l, "\partial"]
\end{tikzcd}$$

Proof. Proposition 12 □

Definition 42 (homotopy equivalence, homotopy type). *A map $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that $1_Y \simeq fg : Y \to Y$ and $1_X \simeq gf : X \to X$. If there exists a homotopy equivalence between X and Y, X and Y are said to be of the same homotopy type.*

Definition 43 (contractible). *X is contractible if $X \to *$ is a homotopy equivalence*

Corollary 3. *Any homotopy equivalence induces an isomorphism in homology*

Proof. As $fg \simeq 1_Y$, then $H_n(f)H_n(g) = 1_{H_n(Y)}$. Similarly, $H_n(g)H_n(f) = 1_{H_n(X)}$. Therefore, $H_n(f)$ is an isomorphism □

2.1.6 LOCALITY OF $H_*(-)$

Let $\mathcal{U}$ be an open cover of a topological space X.

Definition 44 ($\mathcal{U}$-small). *A singular simplex $\sigma : \Delta^n \to X$ is $\mathcal{U}$-small if its image lies entirely in a single open set $U \in \mathcal{U}$. Let $C_n^{\mathcal{U}}(X)$ denote the subgroup of $C_n(X)$ generated by $\mathcal{U}$-small singular simplicies, namely $\mathcal{U}$-small n-chains*

$$\begin{tikzcd}
C_n^{\mathcal{U}}(X) \arrow[r, hook] \arrow[d, dashed, "\partial"] & C_n(X) \arrow[d, "\partial"] \\
C_{n-1}^{\mathcal{U}}(X) \arrow[r, hook] & C_{n-1}(X)
\end{tikzcd}$$

The boundary map, homology on $\mathcal{U}$-small chains is defined accordingly.

Theorem 5 (locality principle). *The inclusion map $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence*

Proof. TODO □

Corollary 4. *$H_n(C_n^{\mathcal{U}}(X)) = H_n(C_n(X)) = H_n(X)$*

Proof. chain homotopy equivalence induces isomorphism in homology □

2.1.7 LONG EXACT SEQUENCE IN HOMOLOGY

Special case: $\mathcal{U} = \{U, V\}$, $X = U \cup V$, U, V open. Define u, v, u', v' being the inclusion maps.

$$\begin{tikzcd} & U \arrow[dr, "u"] & \\ U \cap V \arrow[ur, "u'"] \arrow[dr, "v'"'] & & X \\ & V \arrow[ur, "v"'] & \end{tikzcd}$$

Proposition 14. *The diagram below is a short exact sequence*

$$\begin{tikzcd} C_n(U \cap V) \arrow[r, "i=\begin{bmatrix} u'_\# \\ -v'_\# \end{bmatrix}"] & C_n(U) \oplus C_n(V) \arrow[r, "p=\begin{bmatrix} u_\#, v_\# \end{bmatrix}"] \arrow[dr] & C_n^{\mathcal{U}}(X) \arrow[d, "\subseteq"] \\ & & C_n(X) \end{tikzcd}$$

where $i = c \mapsto (c, -c)$ *and* $p = (c_1, c_2) \mapsto c_1 + c_2$

Theorem 6 (Mayer-Vietoris sequence)**.** *Special case:* $\mathcal{U} = \{U, V\}$, $X = U \cup V$, U, V *open. There exists a long exact sequence*

$$\begin{tikzcd} & & ... \arrow[dll, "\partial"'] \\ H_n(U \cap V) \arrow[r, "i_*"] & H_n(U) \oplus H_n(V) \arrow[r, "p_*"] & H_n(C_n^{\mathcal{U}}(X)) \arrow[dll, "\partial"'] \\ H_{n-1}(U \cap V) \arrow[r, "i_*"] & H_{n-1}(U) \oplus H_{n-1}(V) \arrow[r, "p_*"] & H_{n-1}(C_{n-1}^{\mathcal{U}}(X)) \arrow[dll, "\partial"'] \\ ... & & \end{tikzcd}$$

2.1.8 PROOF OF LOCALITY PRINCIPLE

barycentric subdivision

Definition 45 (cone operator)**.** *Given a star-shaped subset* D *of* $\mathbb{R}^n$ *(there exists a point* p *such that the line segment from* p *to any point* $x \in D$ *belongs to* D*). The cone operator* $c_p : C_n(D) \to C_{n+1}(D)$ *is defined by a linear extension of* $c_p : S_n(D) \to C_{n+1}(D)$*. Let* $\sigma \in S_n(D)$*, then* $c_p(\sigma)$ *is defined by*

$$c_p(\sigma) : (x_0, x_1, ..., x_{n+1}) \mapsto x_0 p + (1 - x_0)\sigma(\overline{x_1}, ..., \overline{x_{n+1}})$$

where $\overline{x_i} = \frac{x_i}{x_1 + ... + x_{n+1}}$

Definition 46 (subdivision operator)**.** *The subdivision operator* $\mathrm{Sd}_n : C_n(\Delta^n) \to C_n(\Delta^n)$ *is defined by*

$$\mathrm{Sd}_0 = 1$$
$$\mathrm{Sd}_n = c_{b(n)} \, \mathrm{Sd}_{n-1} \, \partial$$

where $c_{b(n)}$ *is the cone operator with respective to the barycentre of* Δ^n

Lemma 4. *Given a chain* $s \in C_n(\Delta^p)$*, we can define a unique operator, a linear transformation*

$$S^\bullet : C_p(\bullet) \to C_n(\bullet)$$

such that

1. $S^{\Delta^p}(1) = s$ *where* $1 : \Delta^p \mapsto \Delta^p$ *is the identity simplex*
2. *for any map* $f : X \to Y$*, the diagram below commutes*

$$\begin{tikzcd} C_p(X) \arrow[r, "S^X"] \arrow[d, "f_\#"'] & C_n(X) \arrow[d, "f_\#"'] \\ C_p(Y) \arrow[r, "S^Y"'] & C_n(Y) \end{tikzcd}$$

Proof. (Uniqueness of S^X)

$$\begin{tikzcd}
C_p(\Delta^p) \arrow[r, "S^{\Delta^p}"] \arrow[d, "\sigma_\#"'] & C_n(\Delta^p) \arrow[d, "\sigma_\#"'] \\
C_p(X) \arrow[r, "S^X"'] & C_n(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, mapsto, "S^{\Delta^p}"] \arrow[d, mapsto, "\sigma_\#"] & s \arrow[d, mapsto, "f_\#"] \\
\sigma \arrow[r, mapsto, "S^X"] & S^X\sigma_\# 1 = \sigma_\# S^{\Delta^p} 1
\end{tikzcd}$$

Let $\sigma : \Delta^n \to X$, the commutativity implies the unique definition of S^X by $S^X\sigma = S^X\sigma_\# 1 = \sigma_\# S^{\Delta^p} 1 = \sigma_\# s$ □

Definition 47 (subdivision operator)**.** *Given* $\operatorname{Sd}_n : C_n(\Delta^n) \to C_n(\Delta^n)$*, by Lemma 4, naturality of* S^X *induces a unique map* $\operatorname{Sd}_n : C_n(X) \to C_n(X)$ *such that for all* $\sigma : \Delta^n \to X$, $\operatorname{Sd}_n \sigma = \operatorname{Sd}_n \sigma_\# 1 = \sigma_\# \operatorname{Sd}^n 1$ *where* $1 : \Delta^n \to \Delta^n$ *is the identity simplex.*

Proposition 15 (boundary of cone operator)**.** *Given the cone operator* $c_p : C_n(D) \to C_{n+1}(D)$ *then its boundary is*

$$\partial c_p = \begin{cases} 1 - c_p\partial, & n > 0 \\ 1 - p\epsilon, & n = 0 \end{cases}$$

where $\epsilon : C_0(X) \to \mathbb{Z}$ *is the augmentation map*

Proposition 16. $Sd_n : C_n(X) \to C_n(X)$ *is a chain map, that is* $\operatorname{Sd}_n \partial = \partial \operatorname{Sd}_{n-1}$

apply formula

Proposition 17. Sd *is chain homotopic to* 1

Proof.

Let defined $T_n : C_n(X) \to C_{n+1}(X)$ as follows:

$$\begin{aligned}
T_0 &= 0 \\
T_n &= c_{b(n)}(1 - \operatorname{Sd}_n - T_{n-1}\partial)
\end{aligned}$$

We will prove that $\partial T_n = 1 - \operatorname{Sd}_n - T_{n-1}\partial$ by induction. Base case: $n = 1$, then

$$\begin{aligned}
\partial T_1 &= \partial c_{b(1)}(1 - \operatorname{Sd}_1 - T_0\partial) \\
&= (1 - c_{b(1)}\partial)(1 - \operatorname{Sd}_1) \\
&= 1 - \operatorname{Sd}_1 - c_{b(1)}\partial + c_{b(1)}\partial \operatorname{Sd}_1 \\
&= 1 - \operatorname{Sd}_1 - c_{b(1)}\partial + c_{b(1)} \operatorname{Sd}_0 \partial \\
&= 1 - \operatorname{Sd}_1 - c_{b(1)}\partial + c_{b(1)}\partial \\
&= 1 - \operatorname{Sd}_1 \\
&= 1 - \operatorname{Sd}_1 - T_0
\end{aligned}$$

Induction case, suppose $\partial T_{n-1} = 1 - \operatorname{Sd}_{n-1} - T_{n-2}\partial$, then

$$\begin{aligned}
\partial T_n &= (1 - c_p\partial)(1 - \operatorname{Sd}_n - T_{n-1}\partial) \\
&= (1 - \operatorname{Sd}_n - T_{n-1}\partial) - c_p(\partial - \partial \operatorname{Sd}_n - \partial T_{n-1}\partial) \\
&= (1 - \operatorname{Sd}_n - T_{n-1}\partial) - c_p(\partial - \partial \operatorname{Sd}_n - (1 - \operatorname{Sd}_{n-1} - T_{n-2}\partial)\partial) \\
&= 1 - \operatorname{Sd}_n - T_{n-1}\partial
\end{aligned}$$

□

Proposition 18. $\operatorname{Sd}^k : C_\bullet(X) \to C_\bullet(X)$ *is a chain map*

Proof. Let g be a chain map, then $\partial gg = g\partial g = gg\partial$ □

Corollary 5. 1 *and* Sd^k *are chain homotopic*

Proof. Let $G = 1 + \operatorname{Sd} + \operatorname{Sd}^2 + ... + \operatorname{Sd}^{k-1}$, then

$$1 - \operatorname{Sd}^k = G(1 - \operatorname{Sd}) = G(\partial T + T\partial) = \partial(GT) + (GT)\partial$$

□

Theorem 7 (locality principle)**.**

Proof.

TODO - assignment 1 □

2.1.9 RELATIVE HOMOLOGY

Definition 48 (pair, quotient chain complex, relative homology). *A pair (X, A) is a space X together with a subspace $A \subseteq X$*

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, hook] \arrow[d, "\partial"] & C_n(X) \arrow[r, two heads] \arrow[d, "\partial"] & C_n(X)/C_n(A) \arrow[r] \arrow[d, dashed, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, hook] & C_{n-1}(X) \arrow[r, two heads] & C_{n-1}(X)/C_{n-1}(A) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r, hook] & c + a \arrow[r, two heads] \arrow[d, "\partial"] & {[c]} \arrow[r] \arrow[d, dashed, "\partial"] & 0 \\
\ldots \arrow[r, hook] & \partial c + \partial a \arrow[r, two heads] & {[\partial c]} \arrow[r] & 0
\end{tikzcd}$$

The boundary map on quotient chain complex $C_n(X)/C_n(A)$ is induced from the boundary map on chain complex $C_n(X)$ such that the diagram remains commutative. The quotient chain complex is also denoted by $C_n(X, A) = C_n(X)/C_n(A)$. The relative homology is defined by the homology on $C_n(X, A)$ and denoted by $H_n(X, A) = H_n(C_n(X, A))$. The short exact sequence $C_n(A) \xhookrightarrow{i} C_n(X) \xrightarrow{p} C_n(X)/C_n(A)$ induces a long exact sequence

$$\begin{tikzcd}
 & & \ldots \arrow[dll, "\partial"'] \\
H_n(A) \arrow[r, "i"] & H_n(X) \arrow[r, two heads, "p"] & H_n(X, A) \arrow[dll, "\partial"'] \\
H_{n-1}(A) \arrow[r, "i"] & H_{n-1}(X) \arrow[r, two heads, "p"] & H_{n-1}(X, A) \arrow[dll, "\partial"'] \\
\ldots & &
\end{tikzcd}$$

Remark 15. *$C_n(X, \varnothing) = C_n(X)$, then $H_n(X, \varnothing) = H_n(X)$*

Definition 49 (map of pairs). *$f : (X, A) \to (Y, B)$ is a map of pairs of $f : X \to Y$ and $f(A) \subseteq B$*

Definition 50 (naturality of induced map in pair). *Let $f : (X, A) \to (Y, B)$, the induced map $f_\# : C_n(X, A) \to C_n(Y, B)$ is well-defined since the map $f_\# : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$. The naturality is followed by naturality in absolute chains*

$$\begin{tikzcd}
C_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X, A) \arrow[d, "f_\#"] \\
C_n(Y, B) \arrow[r, "\partial"] & C_{n-1}(Y, B)
\end{tikzcd}$$

Theorem 8 (excision). *Let (X, A) be a pair and $U \subseteq X$ such that $\overline{U} \subseteq \operatorname{int} A$. Then,*

$$H_n(X - U, A - U) \cong H_n(X, A)$$

Proof.
TODO ☐

Proof.
TODO ☐

Lemma 5 (five lemma). *Given $A \to B \to C \to D \to E$ and $A' \to B' \to C' \to D' \to E'$ exact. Then*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

$C \to C'$ is a monomorphism if

- *$D \to D'$ is a monomorphism*
- *$B \to B'$ is a monomorphism*
- *$A \to A'$ is a epimorphism*

$C \to C'$ is a epimorphism if

- *$B \to B'$ is a epimorphism*
- *$D \to D'$ is a epimorphism*
- *$E \to E'$ is a monomorphism*

Proof.
TODO □

Definition 51 (pointed space, reduced homology)**.** *A pointed space is a pair $(X, *)$ such that $*$ is a single point. The reduced homology is a pointed space is denoted by $\overline{H}_n(X) = H_n(X, *)$*

Proposition 19. *Some propositions on reduced homology*

- *$H_0(X) = \overline{H}_0(X) \oplus \mathbb{Z}$*
- *$H_n(X) = \overline{H}_n(X)$ for $n \geq 1$*

2.1.10 EXPRESSING RELATIVE HOMOLOGY IN TERMS OF REDUCE HOMOLOGY

Definition 52 (cone)**.** *Let A be a topological space, the cone on A is defined by*

$$CA = A \times [0,1]/A \times \{0\}$$

Note: $A/\varnothing = A \amalg *$, then $C\varnothing = *$

Lemma 6. *The cone on any space A is contractible, that is $CA \simeq *$. In other words, any space can be embedded into a contractible space.*

Proof.
TODO □

Definition 53 (cone of the pair)**.** *The cone of the pair (X, A) is*

$$X \cup_A CA = X \amalg CA/a \sim (a,1)$$

where $a \sim (a,1)$ denotes the equivalence relation between point $a \in A$ and point $(a,1) \in CA$

Proposition 20. $H_n(X, A) = \overline{H}_n(X \cup_A CA)$

Proof.
TODO □

Definition 54 (suspension)**.** *The cone of pair (CA, A), denoted by ΣA*

2.2 CELLULAR HOMOLOGY

closure-finite weak topology

Definition 55 (cell complex)**.** *A cell complex is a space X with a filtration $X_0 \subseteq X_1 \subseteq ... \subseteq X$ by subspaces such that*

1. *X_0 is discrete*
2. *for each n, X_n is the pushout of the diagram below*

$$\begin{tikzcd} \coprod_{\alpha \in \mathcal{A}_n} \partial D^n \arrow[r, hook] \arrow[d] & \coprod_{\alpha \in \mathcal{A}_n} D^n \arrow[d] \\ X_{n-1} \arrow[r, hook] & X_n \end{tikzcd}$$

   where D^n is a space homeomorphic to an n-dimensional disk, ∂D^n is the boundary of D^n (S^{n-1}). X_n is called the n-skeleton of X, $\coprod \partial D^n \to X_{n-1}$ is called the attaching map, $\coprod D^n$ is called the characteristic map.

3. *$X = \varinjlim X_n$*

Remark 16. *Some remarks on cell complex*

- $\mathbb{R}P^{n-1}$ *is a cell complex with* $(\mathbb{R}P^{n-1})_k = \mathbb{R}P^k \subseteq \mathbb{R}P^n$. $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n = \varinjlim \mathbb{R}P^n$
- S^n *is a cell complex*
- *Grassmannians,* $\mathrm{Gr}_k(\mathbb{R}^n)$*, the set of* k*-dimensional subspace of* $\mathbb{R}^n$*, is a cell complex*

Definition 56 (finite type, finite dimensional)**.** *A cell complex* $X_0 \subseteq X_1 \subseteq ... \subseteq X$ *is of finite type if* X_0 *is finite and each* $\mathcal{A}_n$ *is finite (*$|X_0|$ *is the number of* 0*-cells,* $|\mathcal{A}_n|$ *is the number of* n*-cells).* X *is of finite dimensional if* $\mathcal{A}_n = \varnothing$ *for* $n >> 0$*.* X *is of finite if it is both finite type and finite dimensional.*

2.2.1 THE CONSTRUCTION OF CELLULAR HOMOLOGY

Let X_0 be the pushout of the diagram below

$$\begin{tikzcd} \coprod_{\alpha\in\mathcal{A}_0} \partial D^0 = \varnothing \arrow[r, hook] \arrow[d] & \coprod_{\alpha\in\mathcal{A}_0} D^0 \arrow[d] \\ X_{-1} = \varnothing \arrow[r, hook] & X_0 \end{tikzcd}$$

Then, from the definition of cell complex,

$$X_n/X_{n-1} = \coprod_{\alpha\in\mathcal{A}_n} D^n \Big/ \coprod_{\alpha\in\mathcal{A}_n} \partial D^n = \bigvee_{\alpha\in\mathcal{A}_n} S^n$$

where $S^n = D^n/\partial D^n$ (why do the two quotient topologies coincide? think about it. X_n/X_{n-1} makes X_{n-1} to be one point). The wedge sum $\vee$ is defined by follows: Let $(A, *_A), (B, *_B)$ be pointed spaces, then

$$A \vee B = A \amalg B/*_A \sim *_B$$

Remark 17. *Wedge sum is the coproduct in the category of pointed spaces*

Let H be a homology functor satisfying Eilenberg-Steenrod axioms. Given $A \subseteq X$, as CA contractible, $X \cup_A CA \simeq X/A$, then $\overline{H}_q(X \cup_A CA) \cong \overline{H}_q(X/A)$. Therefore, $H_q(X, A) = \overline{H}_q(X/A)$

$$H_q(X, A) \xrightarrow{\cong} \overline{H}_q(X \cup_A CA) \xrightarrow{\cong} \overline{H}_q(X/A)$$

Definition 57 (cofibration)**.** *A map* $f : A \to X$ *is a cofibration if the homotopy* $H_A : A \times I \to W$ *can be extended to* $H_X : X \times I \to W$*, namely, homotopy extension property. That is, given the diagram below, there exists a map* $H_X : X \times I \to W$ *such that the diagram below commutes*

$$\begin{tikzcd} & & W \\ X \arrow[urr, "g"] \arrow[r, "i_1"'] & X \times I \arrow[ur, dashed, "H_X"] & \\ A \arrow[u, "f"] \arrow[r, "i_1"'] & A \times I \arrow[u, "f\times 1"'] \arrow[uur, "H_A"'] & \end{tikzcd}$$

where i_1 *denotes the map* $x \mapsto (x, 1)$

Proof. Let $Z = X \cup_A (A \times I)$ be the pushout of $X \xleftarrow[f]{} A \xrightarrow[i_1]{} A \times I$ where $X \cup_A (A \times I) = X \amalg (A \times I)/\sim$ with $f(a) \sim (a, 1)$ for all $a \in A$. There exists a retraction $r : X \times I \to Z$. Define $H = hr$ where $h : Z \to W$ is the unique map from pushout Z to W $\square$

Lemma 7. *Some lemmas on cofibration*

1. *the composite of cofibrations is a cofibration*
2. *the coproduct of (closed) cofibrations is a (closed) cofibration*
3. *(closed) cofibration is closed under cobase change, that is, if* $f : A \to X$ *is a (closed) cofibration and* $Y = X \cup_A B$ *is the pushout of* $B \xleftarrow{g} A \xrightarrow{f} X$ *, then* $\overline{f} : B \to Y$ *is a (closed) cofibration*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ B \arrow[r, "\overline{f}"] & Y \end{tikzcd}$$

where $X \cup_A B = X \amalg B / \sim$ with $f(a) \sim g(a)$ for all $a \in A$

Corollary 6. *$X_q \hookrightarrow X_n$ for $q \leq n$ is a closed cofibration*

Proof. the composite of cofibrations is a cofibration (inclusion map is a cofibration) □

Corollary 7. *$H_q(X_n, X_{n-1}) = \overline{H}_q(X_n/X_{n-1}) = \overline{H}_q(\bigvee_{\alpha \in \mathcal{A}_n} S^n) = \bigoplus_{\alpha \in \mathcal{A}_n} \overline{H}_q(S^n) = \begin{cases} 0, & q \neq n \\ \mathbb{Z}\mathcal{A}_n, & q = n \end{cases}$ where $\mathbb{Z}$ denotes the free abelian group functor, that is, $\mathbb{Z}\mathcal{A}_n$ is the free abelian group generated by $\mathcal{A}_n$*

Proof. In the category of pointed space, morphisms are the continuous maps that fix the basepoint and wedge sum is the coproduct. As H_q is a functor on that category. Let X, Y be pointed spaces, we must have the coproduct of two abelian groups $H_q(X)$ and $H_q(Y)$ being $H_q(X \vee Y)$, that is, $H_q(X) \oplus H_q(Y) = H_q(X \vee Y)$ □

The long exact sequence for (X_n, X_{n-1})

$$\begin{tikzcd} & & H_{q+1}(X_n, X_{n-1}) \arrow[dll, "\partial"'] \\ H_q(X_{n-1}) \arrow[r, "i_*"'] & H_q(X_n) \arrow[r] & H_q(X_n, X_{n-1}) \end{tikzcd}$$

the map $i_* : H_q(X_{n-1}) \to H_q(X_n)$ is an isomorphism unless $q = n, n-1$. This is due to $H_q(X_n, X_{n-1}) = 0$ and $H_{q+1}(X_n, X_{n-1})$ otherwise. Therefore,

Corollary 8. *$H_q(X_n) \cong H_q(X_{n-1}) \cong ... \cong H_q(X_0) = 0$ for $q > n \geq 0$. That is, homology groups do not see the skeleton of dimension less than its.*

Corollary 9. *$H_q(X_n) \cong H_q(X_{n+1}) \cong ... \cong H_q(X)$ for $q < n$. That is, homology groups see the same thing on skeleton of dimension more than its.*

When $q = n$, the long exact sequence

$$0 = H_n(X_{n-1}) \xrightarrow[i_*]{} H_q(X_n) \xrightarrow[\partial]{} H_q(X_n, X_{n-1}) = \mathbb{Z}\mathcal{A}_n$$

Then

Corollary 10. *$H_n(X_n)$ is free abelian (a subgroup of $H_n(X_n, X_{n-1}) = \mathbb{Z}\mathcal{A}_n$). Moreover, the number of generators of $H_n(X_n)$ is bounded above by number of n-cells.*

Now using the lemmas, we put the long exact sequence of the pairs $(X_{n+1}, X_n), (X_n, X_{n-1})$ together,

$$\begin{tikzcd} ... \arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial_{n+1}"'] \arrow[drr, dashed, "d_n = i_n \partial_{n+1}"] & & & \\ 0 = H_n(X_{n-1}) \arrow[r] & H_n(X_n) \arrow[rr, "i_n"] \arrow[d] & & H_n(X_n, X_{n-1}) \arrow[d, "\partial_n"'] \arrow[dr, dashed, "d_{n-1} = i_{n-1}\partial_n"] & \\ & H_n(X_{n+1}) = H_n(X) \arrow[d] & 0 = H_{n-1}(X_{n-2}) \arrow[r] & H_{n-1}(X_{n-1}) \arrow[r, "i_{n-1}"] \arrow[d] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[d] \\ & H_n(X_{n+1}, X_n) = 0 & & ... & ... \end{tikzcd}$$

Definition 58 (cellular chain complex, cellular homology)**.** *The abelian groups $H_n(X_n, X_{n-1})$ and homomorphisms $d_n : H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1})$ form a chain complex and called cellular chain complex of the cell complex $X_0 \subseteq X_1 \subseteq ... \subseteq X$ with respect to homology H. Denote $H_n(X_n, X_{n-1})$ by $C_n^{CW}(X)$. We have*

$$C_n^{CW}(X) = \mathbb{Z}\mathcal{A}_n$$

That is, the number of generators of $C_n^{CW}(X)$ is exactly the number of n-cells. Cellular homology is defined from the cellular chain complex and denoted by $H_n(C_n^{CW}(X)) = H_n^{CW}(X)$.

Theorem 9. *$H_n(X) = H_n^{CW}(X)$ and the isomorphism is also natural in the cell complex.*

Proof.

$$\begin{aligned}
H_n(X) &= H_n(X_{n+1}) \\
&= \operatorname{im}(H_n(X_n) \to H_n(X_{n+1})) && (H_n(X_n) \to H_n(X_{n+1}) \text{ is surjective}) \\
&= \frac{H_n(X_n)}{\ker(H_n(X_n) \to H_n(X_{n+1}))} && (\text{first isomorphism theorem}) \\
&= \frac{H_n(X_n)}{\operatorname{im}\partial_{n+1}} && (\text{exactness}) \\
&= \frac{\operatorname{im} i_n}{\operatorname{im} d_n} && (i_n \text{ is injective}) \\
&= \frac{\ker \partial_n}{\operatorname{im} d_n} && (\text{exactness}) \\
&= \frac{\ker d_{n-1}}{\operatorname{im} d_n} && (i_{n-1} \text{ injective}) \\
&= H_n^{CW}(X)
\end{aligned}$$

□

Theorem 10 (cellular boundary formula)**.** *TODO*

2.2.2 COEFFICIENTS IN AN ABELIAN GROUP

Definition 59. *Let A be an abelian group, cellular chain complex of X is*

$$\begin{aligned}
C_n^{CW}(X;A) &= H_n(X_n, X_{n-1}; A) = H_n(X_n, X_{n-1}) \otimes A = \bigoplus_{\alpha\in\mathcal{A}_n} A = \left(\bigoplus_{\alpha\in\mathcal{A}_n} \mathbb{Z}\right) \otimes A \\
H_n(X;A) &= H_n^{CW}(X;A)
\end{aligned}$$

2.3 HOMOLOGY WITH COEFFICIENTS

2.3.1 TENSOR PRODUCT

Definition 60 (tensor product)**.** *Let A, B, C be abelian groups. A bilinear map $A \times B \to C$ is a function $f : A \times B \to C$ such that*

$$\begin{aligned}
f(a + a', b) &= f(a, b) + f(a', b) \\
f(a, b + b') &= f(a, b) + f(a, b')
\end{aligned}$$

The set of bilinear maps $A \times B \to C$ is denoted by $\operatorname{Bil}(A, B; C)$. There exists an abelian group $A \otimes B$ and bilinear map $A \times B \to A \otimes B$ such that for any C and any bilinear map $g : A \times B \to C$, there exists a unique homomorphism $A \otimes B \to C$ such that the following diagram commutes

$$\begin{tikzcd}
A \otimes B \arrow[r] & C \\
A \times B \arrow[u] \arrow[ur, "g"] &
\end{tikzcd}$$

Moreover, $A \otimes B$ is universal. universal mapping property

Remark 18. *:*

1. *If $A \times B \to D$ is bilinear and $D \to C$ is a homomorphism, then the composite $A \times B \to D \to C$ is bilinear*
2. *Let B be an abelian group, then $\otimes B$, the right tensoring of B, is a functor on the category of abelian groups* Ab
3. *$\mathbb{Z} \otimes B = B$*

Proof.

1. trivial

2. consequence of $A \times B \to A \otimes B$ being a functor
3. define the map $f : Z \times B \to B$ by $f(n, b) = nb$ and prove that B satisfies the tensor product factoring property. Hence, $\mathbb{Z} \otimes B = B$

$\square$

Proposition 21. $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Bil}(A, B; C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$

Proof. $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Bil}(A, B; C)$ is done by the definition of tensor product. $\mathrm{Bil}(A, B; C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$ is done by natural evaluation map. $\square$

Corollary 11. $\mathrm{Hom}((\bigoplus_\alpha A_\alpha) \otimes B, C) = \mathrm{Hom}(\bigoplus_\alpha A_\alpha \otimes B, C)$

Proof.

$$\begin{aligned}
\mathrm{Hom}\left(\left(\bigoplus_\alpha A_\alpha\right) \otimes B, C\right) &= \mathrm{Hom}\left(\bigoplus_\alpha A_\alpha, \mathrm{Hom}(B, C)\right) \\
&= \prod_\alpha \mathrm{Hom}(A_\alpha, \mathrm{Hom}(B, C)) \\
&= \prod_\alpha \mathrm{Hom}(A_\alpha \otimes B, C) \\
&= \mathrm{Hom}\left(\bigoplus_\alpha A_\alpha \otimes B, C\right)
\end{aligned}$$

$\square$

Lemma 8. *Let $\mathcal{C}$ be a category, then $f \in \mathcal{C}(X, Y)$ is an isomorphism in $\mathcal{C}$ if and only if for all $W \in \mathrm{ob}\,\mathcal{C}$, $\mathcal{C}(Y, W) \to \mathcal{C}(X, W)$ defined by $g \mapsto gf$ is bijective*

Proof. By surjectivity, $1 \in C(X, X)$ has a preimage $g \in C(Y, X)$ such that $g \mapsto gf = 1$. Now, $fg \in C(Y, Y)$ is mapped to $g = gfg \in C(X, Y)$. Moreover, $1 \in C(Y, Y)$ is also mapped to $g \in C(X, Y)$. By injectivity, $fg = 1$. $\square$

Proposition 22. *$\beta : \bigoplus(A_\alpha \otimes B) \to (\bigoplus A_\alpha) \otimes B$ is an isomorphism*

Proof. From the corollary above $\square$

Lemma 9. *Suppose $A' \to A \to A'' \to 0$ in* Ab*. Then $A' \to A \to A'' \to 0$ is exact if and only if for all C, $\mathrm{Hom}(A', C) \leftarrow \mathrm{Hom}(A, C) \leftarrow \mathrm{Hom}(A'', C) \leftarrow 0$ is exact.*

Proof. no proof provided $\square$

Corollary 12 (right exactness of tensoring abelian group)**.** *If $A' \to A \to A'' \to 0$ is exact, then $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ is also exact*

Proof.
A sequence of exact sequences

1. $A' \to A \to A'' \to 0$
2. $\mathrm{Hom}(A', \mathrm{Hom}(B, C)) \to \mathrm{Hom}(A, \mathrm{Hom}(B, C)) \to \mathrm{Hom}(A'', \mathrm{Hom}(B, C)) \to 0$
3. $\mathrm{Hom}(A' \otimes B, C) \to \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(A'' \otimes B, C) \to 0$
4. $A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$

$\square$

Proposition 23 (an example of free resolution of abelian group)**.** *Let A be an arbitrary abelian group, if $S \subseteq A$ generates A, then the natural projection $\eta : \mathbb{Z}S \to A$ is a surjection where $\mathbb{Z}$ is the free abelian group functor. Let $F_0 = \mathbb{Z}S$, $F_1 = \ker \eta$ (F_1 is free abelian as it is a subgroup of F_0) and $M : F_1 \to F_0$ be the inclusion map, then the sequence below is short exact*

$$0 \longrightarrow F_1 \xrightarrow{M} F_0 \xrightarrow{\eta} A \longrightarrow 0$$

Let $F = \bigoplus_\alpha \mathbb{Z}$ be a free abelian group, then $F \otimes B = (\bigoplus_\alpha \mathbb{Z}) \otimes B = \bigoplus_\alpha (\mathbb{Z} \otimes B) = \bigoplus_\alpha B$. Therefore, for any abelian group B, the sequence below is also exact

$$\bigoplus_{S_1} B \xrightarrow{M} \bigoplus_{S_0} B \xrightarrow{\eta} A \otimes B \longrightarrow 0$$

Remark 19. *Let $A = \mathbb{Z}/m\mathbb{Z}$, $S = \{1\}$, then* $m\mathbb{Z} \cong \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow A \longrightarrow 0$ *is exact. And*

$$\begin{tikzcd}
\mathbb{Z} \otimes B \arrow[r, "m"] \arrow[d, dash, "="] & \mathbb{Z} \otimes B \arrow[r] \arrow[d, dash, "="] & A \otimes B \arrow[r] \arrow[d, dash, "="] & 0 \\
B \arrow[r, "m"] & B \arrow[r] & \operatorname{coker}(m) = B/mB \arrow[r] & 0
\end{tikzcd}$$

is exact. Therefore, $\mathbb{Z}/m\mathbb{Z} \otimes B = B/mB$

Lemma 10. *There is a natural isomorphism $A \otimes B \cong B \otimes A$*

Proof. short □

Lemma 11 (right exactness of tensoring free abelian group)**.** *If F is a free abelian group and* $A' \longrightarrow A \longrightarrow A''$ *is exact at A, then*

$$A' \otimes F \longrightarrow A \otimes F \longrightarrow A'' \otimes F$$

is exact at $A \otimes F$

Proof. As F is free, write $F = \bigoplus_\alpha \mathbb{Z}$, then $A \otimes F = A \otimes \bigoplus_\alpha \mathbb{Z} = \bigoplus_\alpha A$ □

Proposition 24. *:*

1. *In the category* Ch × Ab *where* Ch *is the category of chain complexes and* Ab *is the category of abelian groups, then there is a natural transformation α*

$$\begin{tikzcd}
H_n(C_\bullet) \otimes A \arrow[r, "\alpha_{C_\bullet, A}"] \arrow[d] & H_n(C_\bullet \otimes A) \arrow[d] \\
H_n(C'_\bullet) \otimes A' \arrow[r, "\alpha_{C'_\bullet, A'}"] & H_n(C'_\bullet \otimes A')
\end{tikzcd}$$

   where $\alpha : H_n(C_\bullet) \times A \to H_n(C_\bullet \otimes A)$ is defined by a bilinear map as follows

$$[z], a \mapsto [z \otimes a]$$

   for $z \in C_\bullet, a \in A$.

2. *α is compatible with direct sum*

$$\begin{tikzcd}
H_n(C_\bullet) \otimes (\bigoplus_\alpha A_\alpha) \arrow[r] \arrow[d, dash, "="] & H_n(C_\bullet \otimes (\bigoplus_\alpha A_\alpha)) \arrow[d, dash, "="] \\
\bigoplus_\alpha H_n(C_\bullet) \otimes A_\alpha \arrow[r] & \bigoplus_\alpha H_n(C_\bullet \otimes A_\alpha)
\end{tikzcd}$$

3. *if A is free abelian then α is an isomorphism. An example when α is not an isomorphism:*

$$\begin{array}{ccccc}
 & n-1 & n & n+1 & n+2 \\
C_\bullet & 0 \longleftarrow & \mathbb{Z} \xleftarrow{2} & \mathbb{Z} \longleftarrow & 0 \\
H(C_\bullet) & & \mathbb{Z}/2 & 0 & \\
H(C_\bullet) \otimes \mathbb{Z}/2 & & \mathbb{Z}/2 & 0 & \\
C_\bullet \otimes \mathbb{Z}/2 & 0 \longleftarrow & \mathbb{Z}/2 \xleftarrow{0} & \mathbb{Z}/2 \longleftarrow & 0 \\
H(C_\bullet \otimes \mathbb{Z}/2) & & \mathbb{Z}/2 & \mathbb{Z}/2 &
\end{array}$$

Proof. ($\alpha : H_n(C_\bullet) \otimes A \to H_n(C_\bullet \otimes A)$ is mono)

Let $C_\bullet$ be a chain complex and A be an abelian group, the natural projection $F_0 = \mathbb{Z}S \to A$ where $S \subseteq A$ generates A, $F_1 = \ker(F_0 \to A)$. Suppose $C_\bullet$ is a free abelian group, we have two short exact sequences

$$0 \longrightarrow F_1 \xrightarrow{\quad d \quad} F_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow C_\bullet \otimes F_1 \xrightarrow{1\otimes d} C_\bullet \otimes F_0 \longrightarrow C_\bullet \otimes A \longrightarrow 0$$

Define the natural boundary maps to make $C_\bullet \otimes F_1, C_\bullet \otimes F_0, C_\bullet \otimes A$ chain complexes. The short exact sequence of chain complexes induces a long exact sequence in homology

$$\begin{tikzcd}
 & & \dots \arrow[dll] \\
H_n(C_\bullet \otimes F_1) \arrow[r, "H_n(1\otimes d)"'] & H_n(C_\bullet \otimes F_0) \arrow[r] & H_n(C_\bullet \otimes A) \arrow[dll] \\
H_{n-1}(C_\bullet \otimes F_1) \arrow[r, "H_{n-1}(1\otimes d)"'] & H_{n-1}(C_\bullet \otimes F_0) \arrow[r] & H_{n-1}(C_\bullet \otimes A) \arrow[dll] \\
\dots & &
\end{tikzcd}$$

Hence, $\operatorname{coker} H_n(1 \otimes d) \to H_n(C_\bullet \otimes A)$ is a monomorphism and $H_n(C_\bullet \otimes A) \to \ker H_{n-1}(1 \otimes d)$ is a epimorphism. Hence, the sequence below is exact

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes d) \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \ker H_{n-1}(1 \otimes d) \longrightarrow 0$$

Since $H_n(C_\bullet)$ is abelian, we have another exact sequence

$$H_n(C_\bullet) \otimes F_1 \xrightarrow{1\otimes d} H_n(C_\bullet) \otimes F_0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow 0$$

The diagram below commutes

$$\begin{tikzcd}
H_n(C_\bullet) \otimes F_1 \arrow[r, "1\otimes d"] \arrow[d, "\alpha(\cong)"] & H_n(C_\bullet) \otimes F_0 \arrow[r] \arrow[d, "\alpha(\cong)"] & H_n(C_\bullet) \otimes A \arrow[r] \arrow[d, "\alpha"] & 0 & \\
H_n(C_\bullet \otimes F_1) \arrow[r, "H_n(1\otimes d)"] & H_n(C_\bullet \otimes F_0) \arrow[r] \arrow[d] & H_n(C_\bullet \otimes A) \arrow[r] \arrow[dr] & H_{n-1}(C_\bullet \otimes F_1) \arrow[r, "H_{n-1}(1\otimes d)"] & H_{n-1}(C_\bullet \otimes F_0) \\
 & \operatorname{coker} H_n(1 \otimes d) \arrow[ur] \arrow[uur, dashed] \arrow[d] & & \ker H_{n-1}(1 \otimes d) \arrow[u] \arrow[dr] & \\
0 \arrow[ur] & 0 & & 0 \arrow[u] & 0
\end{tikzcd}$$

We have $H_n(C_\bullet) \otimes A = \operatorname{coker}(1 \otimes d) = \operatorname{coker} H_n(1 \otimes d)$. Since the sequence $0 \to \operatorname{coker} H_n(1 \otimes d) \to H_n(C_\bullet \otimes A) \to \ker H_{n-1}(1 \otimes d) \to 0$ is exact, then $\operatorname{coker} H_n(1 \otimes d) \to H_n(C_\bullet \otimes A)$ is mono. Hence $\alpha : H_n(C_\bullet) \otimes A \to H_n(C_\bullet \otimes A)$ is mono. □

2.3.2 R-MODULE

Definition 61 (R-module)**.** *Given a commutative ring R, an R-module is an abelian group M with bilinear map $R \times M \to M$ such that*

- $1m = m$
- $r(sm) = (rs)m$

for all $r, s \in R, m \in M$

Remark 20. *Some remarks on R-module*

- *R-module is similar to group acting on a set*
- *R-module is a generalization of a vector space where the field of scalars ($\mathbb{R}$ or $\mathbb{C}$) is taken as a commutative ring R*
- *R-module is a generalization of abelian group that is every abelian group is a $\mathbb{Z}$-module.*

Definition 62 (free R-module)**.** *An R-module M is free if there is a subset B, namely basis, such that*

- *B generates M: every element of M can be written as a linear combination of elements in B*
- *elements of B are linearly independent: for every $\{e_1, e_2, ..., e_n\} \subseteq B$, if $r_1e_1 + r_2e_2 + ... + r_ne_n = 0$, then $r_1 = r_2 = ... = r_n = 0$*

Definition 63 (tensor product of R-modules)**.** *Let M, N, L be R-modules and a bilinear map $f : M \times N \to L$ such that*

$$f(rm, n) = rf(m, n) = f(m, rn)$$

Define the tensor product $M \otimes_R N$ with universal property as follows

$$\mathrm{Bil}_R(M, N; L) = \mathrm{Hom}_R(M \otimes_R N, L)$$

The story above works for chain complexes over R-module, that is, the sequence below is exact

$$0 \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

if F_1 is free. In particular, if R is a principal ideal domain (PID) .

2.3.3 FUNDAMENTAL THEOREM OF HOMOLOGICAL ALGEBRA

Definition 64 (resolution, free resolution)**.** *Let $\mathcal{M}$ be the category of R-modules (more generally, abelian category). A resolution of $M \in \mathrm{ob}\,\mathcal{M}$ is a chain complex of R-modules $(F_\bullet, d)$ together with a map $\epsilon : F_0 \to M$ such that the sequence below is exact*

$$0 \longleftarrow M \xleftarrow{\epsilon} F_0 \xleftarrow{d} F_1 \xleftarrow{d} F_2 \xleftarrow{d} ...$$

d is called boundary map, ϵ is called augmentation map. If $F_\bullet$ are free, the sequence is called free resolution of M

Remark 21. *Some remarks on resolution*

- *If we write free resolution as a chain map*

$$\begin{tikzcd} 0 \arrow[d] & F_0 \arrow[l] \arrow[d, "\epsilon"] & F_1 \arrow[l, "d"'] \arrow[d] & F_2 \arrow[l, "d"'] \arrow[d] & ... \arrow[l, "d"'] \\ 0 & M \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & ... \arrow[l] \end{tikzcd}$$

  then ϵ is an isomorphism in homology.

- *If A is a $\mathbb{Z}$-module (abelian group), then $0 \longleftarrow A \xleftarrow{\epsilon} \mathbb{Z}S \xleftarrow{d} \ker\epsilon \longleftarrow 0$ is the free resolution of A where $S \subseteq A$ generates A*

Proposition 25. *Free resolution always exists*

Proof. Denote M by F_{-1}, 0 by F_{-2}. Let $S_n = \ker(F_{n-1} \to F_{n-2})$, construct $F_n = RS_n$ free R-module generated by S_n

$$\begin{tikzcd} F_{n-2} & F_{n-1} \arrow[l] & & F_n = RS_n \arrow[ll, dashed] \\ & & & S_n = \ker(F_{n-1} \to F_{n-2}) \arrow[ull] \end{tikzcd}$$

ask prof why in her proof, M has a generator ∎

Theorem 11 (fundamental theorem of homological algebra, FTHA)**.** *Let $f : M \to M'$ be a map of R-modules and let*

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\epsilon"'] \arrow[d, dashed, "f_0"] & P_1 \arrow[l, "d"'] \arrow[d, dashed, "f_1"] & ... \arrow[l, "d"'] \\ 0 & M' \arrow[l] & P'_0 \arrow[l, "\epsilon'"'] & P'_1 \arrow[l, "d'"'] & ... \arrow[l, "d'"'] \end{tikzcd}$$

be such that P_n is free, the top chain is a chain complex, the bottom chain is exact. Then

1. *for each n, there exists a chain map (lift) $f_n : P_n \to P'_n$ lifting f*
2. *any two lifts are chain homotopic*

Proof.

1. We will prove the first statement by induction.

$$\begin{tikzcd}
 & & K_{n-1} \arrow[dl] \arrow[dd, dashed, "\overline{f_{n-1}}"'] & \\
P_{n-2} \arrow[dd, "f_{n-2}"'] & P_{n-1} \arrow[l] \arrow[dd, "f_{n-1}"'] & & P_n \arrow[ll] \arrow[ul] \arrow[dd, dashed, "f_n"] \\
 & & K'_{n-1} \arrow[dl] & \\
P'_{n-2} & P'_{n-1} \arrow[l] & & P'_n \arrow[ll] \arrow[ul, twoheadrightarrow]
\end{tikzcd}$$

Suppose $f_{n-1} : P_{n-1} \to P'_{n-1}$ and $f_{n-2} : P_{n-2} \to P'_{n-2}$ were constructed. Let $K_{n-1} = \ker(P_{n-1} \to P_{n-2})$ and $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$. Since the both chains are chain complexes, there are maps $P_n \to K_{n-1}$ and $P'_n \to K'_{n-1}$. Moreover, since the bottom chain is exact, the map $P'_n \to K'_{n-1}$ is surjective. We also have the restriction of f_{n-1} on K_{n-1} by $\overline{f_{n-1}} : K_{n-1} \to K'_{n-1}$. This map is well-defined because if $x \in K_{n-1} \subseteq P_{n-1}$, x is mapped to 0 through $P_{n-1} \to P_{n-2} \to P'_{n-2}$, by commutativity, x is also mapped to 0 through $P_{n-1} \to P'_{n-1} \to P'_{n-2}$, that is, x is mapped to K'_{n-1} through f_{n-1}.

Now, since $P'_n \to K'_{n-1}$ is surjective, there exists a map $f_n : P_n \to P'_n$ such that the diagram remains commutative.

For the base case when $n = 0$, let $P_{n-1} = M, P'_{n-1} = M', P_{n-2} = P'_{n-2} = 0, f_{n-1} = f, f_{n-2} = 0$

2. Let $f^{(1)}_\bullet, f^{(2)}_\bullet : P_\bullet \to P'_\bullet$ are two lifts.

$$\begin{tikzcd}
M \arrow[d, "f"'] & P_\bullet \arrow[l, "\epsilon"'] \arrow[d, dashed, bend right, "f^{(1)}_\bullet"'] \arrow[d, dashed, bend left, "f^{(2)}_\bullet"] \\
M' & P'_\bullet \arrow[l, "\epsilon'"]
\end{tikzcd}$$

We will prove that $g_\bullet = f^{(1)}_\bullet - f^{(2)}_\bullet$ is chain homotopic to zero, that is, find map $h_{n+1} : P_n \to P'_{n+1}$ such that $d'h + hd = g$

$$\begin{tikzcd}
0 \arrow[d, "0"'] & P_0 \arrow[l, "d"'] \arrow[d, "g_0"'] & P_1 \arrow[l, "d"'] \arrow[d, "g_1"'] & ... \arrow[l, "d"'] \\
0 & P'_0 \arrow[l, "d'"] & P'_1 \arrow[l, "d'"] & ... \arrow[l, "d'"]
\end{tikzcd}$$

Suppose $h_{n-1} : P_{n-2} \to P'_{n-1}$ and $h_{n-2} : P_{n-3} \to P'_{n-2}$ were constructed

$$\begin{tikzcd}
P_{n-3} \arrow[dr, "h_{n-2}"] & P_{n-2} \arrow[l, "d"'] \arrow[dr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \\
P'_{n-3} & P'_{n-2} \arrow[l, "d'"] & P'_{n-1} \arrow[l, "d'"]
\end{tikzcd}$$

Consider the term below

$$\begin{aligned}
d'(g_{n-1} - h_{n-1}d) &= d'g_{n-1} - d'h_{n-1}d \\
&= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d && \text{(induction)} \\
&= d'g_{n-1} - g_{n-2}d && \text{(top chain is a chain complex)} \\
&= 0 && \text{(commutativity)}
\end{aligned}$$

Hence, there exists a map $P_{n-1} \to K'_{n-1}$ where $K'_{n-1} = \ker(P'_{n-1} \to P'_{n-2})$ such that the diagram below commutes

$$\begin{tikzcd}
P_{n-2} \arrow[ddr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \arrow[dr, dashed] & \\
 & & K'_{n-1} \arrow[dl] \\
P'_{n-2} & P'_{n-1} \arrow[l, "d'"] &
\end{tikzcd}$$

Using the same argument as above, there exists a surjective map $P'_n \to K'_{n-1}$ such that the diagram below commutes. Then, $h_n : P_{n-1} \to P'_n$ is constructed from $P_{n-1} \to K'_{n-1}$ and surjectivity of $P'_n \to K'_{n-1}$.

$$\begin{tikzcd}
P_{n-2} \arrow[ddr, "h_{n-1}"] & P_{n-1} \arrow[l, "d"'] \arrow[dr] \arrow[ddrr, dashed, bend left, "h_n"] & & \\
 & & K'_{n-1} \arrow[dl] & \\
P'_{n-2} & P'_{n-1} \arrow[l, "d'"] & & P'_n \arrow[ll, "d'"] \arrow[ul, dashed, two heads]
\end{tikzcd}$$

For the base case, let $K'_0 = \ker \epsilon'$, the map $M \to M'$ is zero, then $g_0 : P_0 \to P'_0$ is factored through K'_0. The map $P_1 \to K'_0$ is surjective since $M' \leftarrow P'_0 \leftarrow P'_1$ is exact. Hence, $h_1 : P_0 \to P'_1$ can be constructed.

$$\begin{tikzcd}
0 \arrow[dddrr, "h_0=0"] & & P_0 \arrow[ll] \arrow[dl, "\epsilon"'] \arrow[ddd, "g_0"] \arrow[ddr] \arrow[dddrr, dashed, bend left, "h_1"] & & \\
 & M \arrow[ddd, "0"] & & & \\
 & & & K'_0 \arrow[dl] & \\
0 & & P'_0 \arrow[ll] \arrow[dl, "\epsilon'"] & & P'_1 \arrow[ll, "d'"] \arrow[ul, dashed, two heads] \\
 & M' & & &
\end{tikzcd}$$

□

Corollary 13. *A free resolution of M is well-defined as a chain homotopy type.*

Proof. Suppose we have two free resolutions $0 \leftarrow M \leftarrow P_\bullet$ and $0 \leftarrow M \leftarrow P'_\bullet$

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "1"] \arrow[dd, bend right, "1"'] & P_0 \arrow[l] \arrow[d, dashed, "f_0"] \arrow[dd, dashed, bend right, "1"'] & P_1 \arrow[l] \arrow[d, dashed, "f_1"] \arrow[dd, dashed, bend right, "1"'] & \dots \arrow[l] \\
0 & M \arrow[l] \arrow[d, "1"] & P'_0 \arrow[l] \arrow[d, dashed, "g_0"] & P'_1 \arrow[l] \arrow[d, dashed, "g_1"] & \arrow[l] \\
0 & M \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

By FTHA, there are three lifts $1_\bullet : P_\bullet \to P_\bullet, f_\bullet : P_\bullet \to P'_\bullet, g_\bullet : P'_\bullet \to P_\bullet$. As both $1_\bullet$ and $g_\bullet f_\bullet$ are both lifts $P_\bullet \to P_\bullet$. By FTHA, $g_\bullet f_\bullet \simeq 1$. Similarly, $f_\bullet g_\bullet \simeq 1$. Hence, the two free resolutions are of the same chain homotopy type.

TODO - whether free resolution of two R-modules can be chain homotopic?

□

Corollary 14. *There exists a functor* R-Mod $\to$ Ho(Ch(R-Mod)) *where* Ho *denotes the category of homology*

Proof. Given M being an R-module, the functor is defined by the homology of the chain complex from free resolution of M. The functor is well-defined because a free resolution defines a chain homotopy type and chain homotopic chain complexes induces the same homology. Given any map between R-modules M and M', the functor is defined by the chain map between free resolutions of M and M'. The functor is well-defined because two chain homotopic maps induces the same map in homology. □

2.3.4 ADDITIVE FUNCTOR, TOR FUNCTOR $\text{R-Mod} \to \text{Ab}$

Remark 22. *Let $F : \text{R-Mod} \to \text{Ab}$ be a functor from R-module to abelian group. Let $P_\bullet$ be a chain complex of $\mathbb{R}$-module*

$$0 \longleftarrow P_0 \xleftarrow{d} P_1 \xleftarrow{d} \ldots$$

$$0 \longleftarrow FP_0 \xleftarrow{Fd} FP_1 \xleftarrow{Fd} \ldots$$

then,

$$(Fd)(Fd) = F(dd) = F(0)$$

If $F(0) = 0$, then F sends a chain complex to a chain complex.

Definition 65 (additive functor $\text{R-Mod} \to \text{Ab}$)**.** *$F : \text{R-Mod} \to \text{Ab}$ is a additive functor if for any $M, N \in \operatorname{ob}\text{R-Mod}$,*

$$\operatorname{Hom}_R(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is a homomorphism of abelian groups.

Remark 23. *Some remarks on additive functor of* $\text{R-Mod} \to \text{Ab}$

- *in the categories* R-Mod *or* Ab, Hom *is an abelian groups. More generally, they are preadditive categories or* Ab-*enriched categories where* Hom *is equipped with abelian group structure and composition is bilinear.*
- *if $F : \text{R-Mod} \to \text{Ab}$ is additive, then F sends a chain complex to a chain complex, that is, there is an extension* $\text{Ch(R-Mod)} \to \text{Ch(Ab)}$
- *tensor product is an additive functor.*

Lemma 12. *If F is additive and h is a chain homotopy $f_0 \simeq f_1 : C_\bullet \to C'_\bullet$ in* R-Mod*, then $F(h)$ is a chain homotopy $F(f_0) \simeq F(f_1) : F(C_\bullet) \to F(C'_\bullet)$*

Proof. Suppose $d'h + hd = f_0 - f_1$ is a chain homotopy. Then

$$\begin{aligned}
F(d')F(h) + F(h)F(d) &= F(d'h) + F(hd) && (F \text{ is a functor})\\
&= F(d'h + hd) && (F \text{ is an additive functor})\\
&= F(f_0 - f_1) \\
&= F(f_0) - F(f_1) && (F \text{ is an additive functor})
\end{aligned}$$

□

Corollary 15. *Chain homotopy type of $F(P_\bullet)$ is well-defined (where $M \leftarrow P_\bullet$ is a free resolution)*

Remark 24. *TODO*

Definition 66 (Tor functor)**.** *Let R be a ring and M be R-modules, define the* Tor *functor as the homology of chain complex $P_\bullet \otimes_R N$*

$$\operatorname{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N)$$

for $M \leftarrow P_\bullet$ is any free resolution.

Remark 25. *Some remarks on* Tor

- $\operatorname{Tor}_0^R(M, N) = M \otimes_R N$
- *Some example of Tor TODO*

Proposition 26. Tor *is well-defined and functorial in both variables, that is, $\operatorname{Tor}_n^R(M, -)$ and $\operatorname{Tor}_n^R(-, N)$ are functors* $\text{R-Mod} \to \text{Ab}$.

2.3.5 UNIVERSAL COEFFICIENT THEOREM

Theorem 12 (universal coefficient theorem). *Let R be a PID and N be an R-module, then for any chain complex $C_\bullet$, then there is a short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes_R N \xrightarrow{\alpha} H_n(C_\bullet \otimes_R N) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

Proof. Since R is a PID, N admits a free resolution

$$0 \longleftarrow N \longleftarrow P_0 \xleftarrow{d} P_1 \longleftarrow 0$$

By exactness of tensor product, the sequence below is exact

$$0 \longleftarrow C_\bullet \otimes_R N \longleftarrow C_\bullet \otimes_R P_0 \xleftarrow{1\otimes d} C_\bullet \otimes_R P_1 \longleftarrow 0$$

The short exact sequence induces a long exact sequence

$$H_n(C_\bullet \otimes_R P_1) \xrightarrow{H_n(1\otimes_R d)} H_n(C_\bullet \otimes_R P_0) \longrightarrow H_n(C_\bullet \otimes_R N) \longrightarrow H_{n-1}(C_\bullet \otimes_R P_1) \xrightarrow{H_{n-1}(1\otimes_R d)} H_{n-1}(C_\bullet \otimes_R P_0)$$

That induces a short exact sequence

$$0 \longrightarrow \operatorname{coker} H_n(1 \otimes_R d) \longrightarrow H_n(C_\bullet \otimes_R N) \longrightarrow \ker H_{n-1}(1 \otimes_R d) \longrightarrow 0$$

As discussed earlier, $\operatorname{coker} H_n(1 \otimes_R d) = H_n(C_\bullet) \otimes_R N$. Furthermore, we can identify $\ker H_n(1 \otimes_R d)$ by $\ker(1 \otimes_R d)$

$$\begin{tikzcd}
H_{n-1}(C_\bullet) \otimes_R P_1 \arrow[r, "1\otimes_R d"] \arrow[d, "\alpha(\cong)"] & H_{n-1}(C_\bullet) \otimes_R P_0 \arrow[d, "\alpha(\cong)"] \\
H_{n-1}(C_\bullet \otimes_R P_1) \arrow[r, "H_{n-1}(1\otimes_R d)"] & H_{n-1}(C_\bullet \otimes_R P_0)
\end{tikzcd}$$

And $\ker(1 \otimes_R d) = \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), N)$ by exactness of tensor product, the sequence below is exact

$$0 \longleftarrow H_{n-1}(C_\bullet) \otimes_R N \longleftarrow H_{n-1}(C_\bullet) \otimes_R P_0 \xleftarrow{1\otimes d} H_{n-1}(C_\bullet) \otimes_R P_1 \longleftarrow 0$$

□

Remark 26. *Let $C_\bullet = C_\bullet(X; R) = C_\bullet(X) \otimes_{\mathbb{Z}} R$, then*

$$0 \longrightarrow H_n(X; R) \otimes_R N \xrightarrow{\alpha} H_n(X; N) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(X; R), N) \longrightarrow 0$$

2.3.6 KÜNNETH THEOREM

Definition 67 (tensor product of chain complexes). *Let $C_\bullet, D_\bullet$ be chain complexes of R-module. Define the tensor product $C_\bullet \otimes D_\bullet$ by*

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and define the boundary map $\partial : (C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ by a extension of $\partial : C_p \otimes D_q \to (C_\bullet \otimes D_\bullet)_{n-1}$

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$$

where $x \in C_p, y \in D_q$

Theorem 13 (Eilenberg-Zilber theorem). *The Alexander-Whitney (AW) map and Eilenberg-Zilber (EZ) map are natural transformation of functors $\mathrm{Top}^2 \to \mathrm{Ch}(\mathrm{Ab})$ which are naturally chain homotopy inverses*

$$\begin{tikzcd}
C_\bullet(X \times Y) \arrow[r, bend left, "AW"] & C_\bullet(X) \otimes C_\bullet(Y) \arrow[l, bend left, "EZ"]
\end{tikzcd}$$

TODO: method of proof

Remark 27. *Two chain complexes of functors* $\mathrm{Top}^2 \to \mathrm{Ab}$

$$(X, Y) \mapsto C_n(X \times Y)$$
$$(X, Y) \mapsto (C_\bullet(X) \otimes C_\bullet(Y))_n$$

Theorem 14. *There is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes_R H_q(D_\bullet) \longrightarrow H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

if R is a PID, $C_\bullet$ degree-wise free chain complex and this sequence splits

Proof. TODO □

Corollary 16. *Let R be a PID and $C'_\bullet, C_\bullet$ degree-wise R-free. If $C'_\bullet \to C_\bullet$ and $D'_\bullet \to D_\bullet$ are homology isomorphisms, so is $C'_\bullet \otimes D'_\bullet \to C_\bullet \otimes D_\bullet$*

Chapter 3

OUT OF PLACE DISCUSSION

3.1 (BROUWER) DEGREE OF $f : S^n \to S^n$ ()