

# homological\_algebra

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## 1 CATEGORICAL CONSTRUCTION OF HOMOLOGICAL ALGEBRA

**Definition 1** (initial object, terminal object, pointed category, zero map, kernel). *Given a category  $C$ , an object  $0$  is initial if for all  $X \in \text{ob } C$ , there is only one map in  $\text{Hom}(0, X)$ , an object  $*$  is terminal if for all  $X \in \text{ob } C$ , there is only one map in  $\text{Hom}(X, *)$ . Category  $C$  is called pointed if it has initial and terminal objects and the unique map  $0 \rightarrow *$  is an isomorphism.*

*If  $C$  is a pointed category, we use the same symbol  $0$  for both initial object and terminal object. There exists a zero map between any two objects  $M, N \in \text{ob } C$ , defined by*

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ & \searrow 0 & \downarrow \\ & & N \end{array}$$

*the composition of  $M \rightarrow 0$  and  $0 \rightarrow N$ . Let  $f : M \rightarrow N$  be a morphism in  $C$ , a kernel of  $f$  is a map  $i : K \rightarrow M$  such that  $fi = 0$  and such map is universal, that is, if  $j : L \rightarrow M$  with  $fj = 0$ , then it factors through  $K$*

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & & \curvearrowleft & \\ K & \xrightarrow{i} & M & \xrightarrow{f} & N \\ & \uparrow \text{---} & \nearrow j & & \\ & L & & & \\ & \curvearrowleft & & \curvearrowright & \\ & & 0 & & \end{array}$$

*Category  $C$  has kernels if every morphism has a kernel.*

**Definition 2** (preadditive category, Ab-enriched category). *A category  $C$  is called preadditive category (or Ab-enriched category) if for any two objects  $M, N \in \text{ob } C$ ,  $\text{Hom}(M, N)$  is an abelian group and composition is bilinear, that is, if  $f, g, h$  are morphisms in  $C$*

$$\begin{aligned} f(g + h) &= fg + fh \\ (f + g)h &= fh + gh \end{aligned}$$

**Definition 3** (additive category). *A category  $C$  is additive if*

1.  $C$  is pointed
2.  $C$  is preadditive
3.  $C$  admits finite biproduct

*(biproduct) is when product and coproduct coincide*

**Definition 4** (abelian category). *A category  $C$  is abelian if*

1.  $C$  is additive
2. every map in  $C$  has kernel and cokernel
3. every monomorphism in  $C$  is the kernel of its cokernel
4. every epimorphism in  $C$  is the cokernel of its kernel

**Definition 5** (projective class). *Let  $C$  be a pointed category with kernels. A projective class in  $C$  is a pair  $(\mathcal{P}, \mathcal{E})$  where  $\mathcal{P}$  is a collection of objects (called **projectives**) and  $\mathcal{E}$  is a collection of morphisms (called **epimorphisms**) such that*

1. An object  $P$  is **projective** if and only if  $P$  has the universal lifting property against every **epimorphism**  $M \rightarrow N$ , that is, given any **epimorphism**  $M \rightarrow N$ , if there is a map  $P \rightarrow N$ , then it factors through  $M$

$$\begin{array}{ccc} M & \xrightarrow{\text{epi}} & N \\ & \swarrow \text{---} & \uparrow \\ & & P \end{array}$$

2. A morphism  $f : M \rightarrow N$  is an **epimorphism** if and only if every **projective** has the universal lifting property against  $f$ , that is, given any **projective**  $P$ , if there is a map  $P \rightarrow N$ , then it factors through  $M$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \swarrow \text{---} & \uparrow \\ & & P \end{array}$$

3.  $C$  has enough **projectives**, that is, given any object  $M \in \text{ob } C$ , for every **projective**  $P$ , there exists an **epimorphism**  $P \rightarrow M$ .

**Definition 6** (chain complex, acyclic chain complex, exact sequence). In a **pointed category with kernels**, a chain complex is a sequence such that given any subsequence  $A \rightarrow B \rightarrow C$ ,  $A \rightarrow B$  factors through  $\ker(B \rightarrow C)$ , that is, there exists a map  $A \rightarrow \ker(B \rightarrow C)$  such that the diagram below commutes

$$\begin{array}{ccccccc} \dots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow \dots \\ & & \downarrow & & \nearrow & & \\ & & \ker(B \rightarrow C) & & & & \end{array}$$

Equivalently, the composition  $A \rightarrow B \rightarrow C$  is the zero map.

If there is a notion of epimorphism and the map  $A \rightarrow \ker(B \rightarrow C)$  is an epimorphism, then the sequence is called exact at  $B$ . A sequence is called exact sequence or an acyclic chain complex if it is exact everywhere, possibly except the two ends.

**Definition 7** (chain map, chain homotopy). Given two chain complexes  $C_\bullet, D_\bullet$  in a **pointed category with kernels**, for each  $n \in \mathbb{Z}$ , there is a map  $f_n : C_n \rightarrow D_n$  such that the diagram below commutes, then  $f_\bullet$  is called a chain map

$$\begin{array}{ccccccc} \dots & \longleftarrow & C_{n-1} & \longleftarrow & C_n & \longleftarrow & C_{n+1} \longleftarrow \dots \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} \\ \dots & \longleftarrow & D_{n-1} & \longleftarrow & D_n & \longleftarrow & D_{n+1} \longleftarrow \dots \end{array}$$

Chain complexes and chain maps form a category and it is called the category of chain complexes.

Given two chain complexes  $C_\bullet, D_\bullet$  in a **pointed preadditive category with kernels**. Let  $f_\bullet, g_\bullet : C_\bullet \rightarrow D_\bullet$  be two chain maps. A chain homotopy from  $f_\bullet$  to  $g_\bullet$  is a collection of maps  $h_n : C_{n-1} \rightarrow D_n$  such that  $\partial h_{n+1} + h_n \partial = f_n - g_n$

$$\begin{array}{ccccccc} \dots & \xleftarrow{\partial} & C_{n-1} & \xleftarrow{\partial} & C_n & \xleftarrow{\partial} & C_{n+1} \xleftarrow{\partial} \dots \\ & & \searrow h_n & & \searrow h_{n+1} & & \\ \dots & \xleftarrow{\partial} & D_{n-1} & \xleftarrow{\partial} & D_n & \xleftarrow{\partial} & D_{n+1} \xleftarrow{\partial} \dots \end{array}$$

**Theorem 1** (fundamental theorem of homological algebra - FTTH). Let  $C$  be a **pointed category with kernels** and  $(\mathcal{P}, C)$  be a projective class in  $C$ . Given  $f : M \rightarrow M'$  in  $C$  and the diagram below

$$\begin{array}{ccccccc} 0 & \longleftarrow & M & \xleftarrow{\epsilon} & P_0 & \xleftarrow{d} & P_1 \xleftarrow{d} \dots \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ 0 & \longleftarrow & M' & \xleftarrow{\epsilon'} & P'_0 & \xleftarrow{d'} & P'_1 \xleftarrow{d'} \dots \end{array}$$

where both chains are chain complexes, the top chain consists of projectives  $P_n$  and the bottom chain is acyclic. Then,

- There exists a chain map defined by  $f_n : P_n \rightarrow P'_n$
- If  $C$  is preadditive, the lift is unique upto chain homotopy.

Proof.

1. The first statement is proved by induction

$$\begin{array}{ccccc}
P_{n-2} & \longleftarrow & P_{n-1} & \longleftarrow & P_n \\
\downarrow f_{n-2} & & \downarrow f_{n-1} & & \downarrow f_n \\
P'_{n-2} & \longleftarrow & P'_{n-1} & \longleftarrow & P'_n \\
& & & \nearrow & \nearrow \\
& & & K'_{n-1} & 
\end{array}$$

Suppose there exist maps  $f_{n-1} : P_{n-1} \rightarrow P'_{n-1}$  and  $f_{n-2} : P_{n-2} \rightarrow P'_{n-2}$ . Let  $K'_{n-1} = \ker(P'_{n-1} \rightarrow P'_{n-2})$ .

Since the bottom chain is acyclic, the map  $P'_n \rightarrow P'_{n-1}$  factors through  $K'_{n-1}$  by an epimorphism.

Since the top chain is a chain complex, the composition  $P_n \rightarrow P_{n-1} \rightarrow P'_{n-1} \rightarrow P'_{n-2}$  equals  $P_n \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow P'_{n-2}$  and equals 0 zero, so  $P_n \rightarrow P_{n-1} \rightarrow P'_{n-1}$  factors through  $K'_{n-1}$

Since  $P_n$  is projective and  $P'_n \rightarrow K'_{n-1}$  is an epimorphism,  $P_n \rightarrow K'_{n-1}$  factors through  $P'_n$  by a map  $f_n : P_n \rightarrow P'_n$

**Base case:**  $n = 0$ , let  $P_{n-1} = M, P'_{n-1} = M', P_{n-2} = 0, P'_{n-2} = 0$  and  $f_{n-1} = f, f_{n-2} = 0$

2. Let  $f_{\bullet}^{(1)}, f_{\bullet}^{(2)} : P_{\bullet} \rightarrow P'_{\bullet}$  be any two lifts from  $f : M \rightarrow M'$

$$\begin{array}{ccc}
M & \longleftarrow & P_{\bullet} \\
f \downarrow & & \downarrow f_{\bullet}^{(1)} \quad \downarrow f_{\bullet}^{(2)} \\
M' & \longleftarrow & P'_{\bullet}
\end{array}$$

We will prove that  $g_{\bullet} = f_{\bullet}^{(1)} - f_{\bullet}^{(2)}$  is chain homotopic to zero, that is to find maps  $h_{n+1} : P_n \rightarrow P'_{n+1}$  such that  $d'h + hd = g$

$$\begin{array}{ccccccc}
0 & \longleftarrow & d & P_0 & \longleftarrow & d & P_1 & \longleftarrow & d & \dots \\
0 \downarrow & & & g_0 \downarrow & & & g_1 \downarrow & & & \\
0 & \longleftarrow & d' & P'_0 & \longleftarrow & d' & P'_1 & \longleftarrow & d' & \dots
\end{array}$$

Suppose there exists map  $h_{n-1} : P_{n-2} \rightarrow P'_{n-1}$  and  $h_{n-2} : P_{n-3} \rightarrow P'_{n-2}$  such that

$$g_{n-2} - h_{n-2}d = d'h_{n-1}$$

$$\begin{array}{ccccc}
P_{n-3} & \longleftarrow & d & P_{n-2} & \longleftarrow & d & P_{n-1} \\
& \searrow & h_{n-2} & & \searrow & h_{n-1} & \\
P'_{n-3} & \longleftarrow & d' & P'_{n-2} & \longleftarrow & d' & P'_{n-1}
\end{array}$$

Consider the map  $g_{n-1} - h_{n-1}d : P_{n-1} \rightarrow P'_{n-1}$ ,

$$\begin{aligned}
d'(g_{n-1} - h_{n-1}d) &= d'g_{n-1} - d'h_{n-1}d && \text{(preadditive)} \\
&= d'g_{n-1} - (g_{n-2} - h_{n-2}d)d && \text{(induction)} \\
&= d'g_{n-1} - g_{n-2}d && \text{(preadditive, } dd = 0) \\
&= 0 && \text{(} g_{\bullet} \text{ is a chain map)}
\end{aligned}$$

Let  $K'_{n-1} = \ker(d' : P'_{n-1} \rightarrow P'_{n-2})$ .

Since the bottom chain is acyclic, the map  $d' : P'_n \rightarrow P'_{n-1}$  factors through  $K'_{n-1}$  by an epimorphism.

$$\begin{array}{ccccc}
P_{n-2} & \longleftarrow & d & P_{n-1} & \\
& \searrow & h_{n-1} & & \searrow & h_n \\
& & & K'_{n-1} & & \\
& & & \downarrow & & \\
P'_{n-2} & \longleftarrow & d' & P'_{n-1} & \longleftarrow & d' & P'_n
\end{array}$$

As  $d'(g_{n-1} - h_{n-1}d) = 0$ ,  $g_{n-1} - h_{n-1}d$  factors through  $K'_{n-1}$ , that is,  $g_{n-1} - h_{n-1}d$  equals the composition  $P_{n-1} \rightarrow K'_{n-1} \rightarrow P'_{n-1}$

Since  $P_{n-1}$  is projective and  $P'_n \rightarrow K'_{n-1}$  is an epimorphism,  $P_{n-1} \rightarrow K'_{n-1}$  factors through  $P'_n$  by a map  $h_n : P_{n-1} \rightarrow P'_n$ , that is, the  $d'h_n$  equals the composition  $P_{n-1} \rightarrow P'_n \rightarrow K'_{n-1} \rightarrow P'_{n-1}$  and equals the composition  $P_{n-1} \rightarrow K'_{n-1} \rightarrow P'_{n-1}$ , hence

$$d'h_n = g_{n-1} - h_{n-1}d$$

**Base case:**  $n = 0$ , let  $P_{n-2} = 0, P'_{n-2} = 0, P_{n-1} = M, P'_{n-1} = M', h_{n-1} = 0$ , then

$$d'(g_{n-1} - h_{n-1}d) = 0 \quad (d' : P'_{n-1} \rightarrow P'_{n-2} \text{ is the zero map } M' \rightarrow 0)$$

□

**Definition 8** (resolution, projective resolution). *Let  $M$  be an object in a **pointed category with kernels**. A resolution of  $M$  is an exact sequence*

$$0 \longleftarrow M \xleftarrow{\epsilon} P_0 \xleftarrow{d} P_1 \xleftarrow{d} \dots$$

If  $P_n$  are projectives in a projective class  $(\mathcal{P}, \mathcal{E})$ , then the sequence is called  $\mathcal{P}$ -projective resolution.

**Corollary 1.** *Let  $M$  be an object in a **pointed preadditive category with kernels**. Any two projective resolutions of  $M$  are of the same chain homotopy type.*

**Definition 9** (additive functor). *Let  $C, D$  be **preadditive categories**, a functor  $F : C \rightarrow D$  is additive if for every  $M, N \in \text{ob } C$ ,*

$$\text{Hom}(M, N) \rightarrow \text{Hom}(F(M), F(N))$$

*is a homomorphism of abelian groups*

**Remark 1.** *Additive functor preserves chain complex. That is, if  $C_\bullet$  is a chain complex, then*

$$\begin{aligned} \dots &\xleftarrow{d} C_{n-1} \xleftarrow{d} C_n \xleftarrow{d} C_{n+1} \xleftarrow{d} \dots \\ \dots &\xleftarrow{Fd} FC_{n-1} \xleftarrow{Fd} FC_n \xleftarrow{Fd} FC_{n+1} \xleftarrow{Fd} \dots \end{aligned}$$

*the bottom sequence is also a chain complex.*

$$(Fd)(Fd) = F(dd) = F(0) = 0$$

**Remark 2.** *Additive functor preserves chain homotopy. That is, if  $f, g : C_\bullet \rightarrow D_\bullet$  are chain homotopic by a chain homotopy  $h$ , then,  $F(h)$  is a chain homotopy from  $Ff$  to  $Fg$*

$$\begin{aligned} (Fd)(Fh) + (Fh)(Fd) &= F(dh) + F(hd) && (F \text{ is a functor}) \\ &= F(dh + hd) && (F \text{ is additive}) \\ &= F(f - g) && (f \simeq g \text{ by } h) \\ &= F(f) - F(g) && (F \text{ is a functor}) \end{aligned}$$

**Remark 3.** *a functor between additive categories is additive if and only if it preserves finite coproducts - Mac Lane's "Categories for the working mathematician"*

*TODO - <https://math.stackexchange.com/questions/793029/do-covariant-functors-preserve-direct-sums>*

**Definition 10** (left derived functor). *Let  $C$  be a **pointed preadditive category with kernels**, a projective class  $(\mathcal{P}, \mathcal{E})$ . Let  $F : C \rightarrow \text{Ab}$  be an additive functor, then the left derived functor  $F$  with respect to  $(\mathcal{P}, \mathcal{E})$  are  $L_n F : C \rightarrow \text{Ab}$  defined by*

$$(L_n F)(X) = H_n(FP_\bullet)$$

*where  $X \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$  is a  $\mathcal{P}$ -projective resolution.*

**Remark 4.** *As  $F$  is additive and any two projective resolutions of an object are chain homotopic, the left derived functor is well-defined.*

*Moreover, left derived functor is functorial **TODO***

**Remark 5** (motivation of left derived functor). *something related to extending  $FA \rightarrow FB \rightarrow FC \rightarrow 0$  to  $\dots \rightarrow L_2FA \rightarrow L_2FB \rightarrow L_2FC \rightarrow L_1FA \rightarrow L_1FB \rightarrow L_1FC \rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0$  so that if  $F$  is an exact functor (both left and right exact) then  $L_1F = 0$ , that is,  $L_1$  measures how far  $F$  from being exact. In case of  $\text{Tor}$ ,  $\text{Tor}_1$  measures how far  $\otimes M$  from being exact, or how far  $M$  from being projective/free*

*Proof.*

can be proved, take a look at my notes □

## 2 UNIVERSAL COEFFICIENT THEOREM FOR CHAIN COMPLEXES OF $R$ -MODULES

**Remark 6** ( $R\text{-Mod}$  is abelian). *If  $R$  is a commutative ring, the category of  $R$ -modules is abelian*

**Remark 7** (projective module, projective class in  $R\text{-Mod}$ ). *In the category  $R\text{-Mod}$ , there is a projective class  $(\mathcal{P}, \mathcal{E})$  defined by epimorphism being surjective homomorphism. Then, the following are equivalent*

1.  $P \in \text{ob } R\text{-Mod}$  is projective
2. Every short exact sequence  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  splits
3.  $P$  is a direct summand of a free  $R$ -module, that is, there exists  $Q \in \text{ob } R\text{-Mod}$  such that  $P \oplus Q$  is a free  $R$ -module.

*Proof. TODO* □

**Definition 11** (Tor functor). *In the category  $R\text{-Mod}$ , define  $\text{Tor}_n : R\text{-Mod} \times R\text{-Mod} \rightarrow \text{Ab}$  such that  $\text{Tor}(-, M)$  is the  $n$ -th left derived functor of  $(- \times M)$*

$$\text{Tor}_n(N, M) = (L_n(- \otimes M))(N) = H_n(P_\bullet \otimes M)$$

where  $N \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$  is a projective resolution of  $N$

**Remark 8** (some properties of  $\text{Tor}$ ).

$$\begin{aligned} \text{Tor}_n(A, B) &= \text{Tor}_n(B, A) \\ \text{Tor}_n(\varinjlim_\alpha A_\alpha, B) &= \varinjlim_\alpha \text{Tor}_n(A_\alpha, B) \\ \text{Tor}_n(P, B) &= 0 \end{aligned} \quad (\text{if } P \text{ is projective})$$

*TODO*

**Definition 12** (direct sum of chain complexes of  $R$ -module - product, coproduct, biproduct). *In the category  $\text{Ch}(R\text{-Mod})$ , the direct sum  $\oplus : \text{Ch}(R\text{-Mod}) \times \text{Ch}(R\text{-Mod}) \rightarrow \text{Ch}(R\text{-Mod})$  is defined as follows:*

$$(C_\bullet \oplus D_\bullet)_n = C_n \oplus D_n$$

where  $C_\bullet, D_\bullet \in \text{ob } \text{Ch}(R\text{-Mod})$  and the boundary map  $\partial : (C_\bullet \oplus D_\bullet)_n \rightarrow (C_\bullet \oplus D_\bullet)_{n-1}$  is defined by

$$\begin{aligned} \partial : (C \oplus D)_n &\rightarrow (C \oplus D)_{n-1} \\ c \oplus d &\mapsto \partial c \oplus \partial d \end{aligned}$$

**Definition 13** (tensor product of chain complexes of  $R$ -module). *In the category  $\text{Ch}(R\text{-Mod})$ , the tensor product  $\otimes : \text{Ch}(R\text{-Mod}) \times \text{Ch}(R\text{-Mod}) \rightarrow \text{Ch}(R\text{-Mod})$  is defined as follows:*

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

where  $C_\bullet, D_\bullet \in \text{ob } \text{Ch}(R\text{-Mod})$  and the boundary map  $\partial : (C_\bullet \otimes D_\bullet)_n \rightarrow (C_\bullet \otimes D_\bullet)_{n-1}$  is the linear extension of  $\partial : C_p \otimes D_q \rightarrow (C_\bullet \otimes D_\bullet)_{n-1}$  where

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d$$

*Proof. TODO - bilinear chain map factors through tensor product* □

**Lemma 1** (on the spot categorical lemma). *Let  $F, G : C \rightarrow D$  be functors that are natural to the identity functor. Let  $f : A \rightarrow B$  be a morphism in  $C$ , if  $FA = GA$  and  $FB = GB$ , then  $Ff = Gf$  if  $C = D$ , then it produces a bunch of commutative squares, easy to prove*

$$\begin{array}{ccc}
A & & FA \xrightarrow{=} GA \\
\downarrow f & & \downarrow Ff \quad \downarrow Gf \\
B & & FB \xrightarrow{=} GB
\end{array}$$

**Theorem 2** (universal coefficient theorem - UCT). *In the category  $R\text{-Mod}$  where  $R$  is a PID. Let  $C_\bullet$  be a chain complex of free  $R$ -modules and  $N$  be an  $R$ -module. There is a natural short exact sequence (the map  $H_n(C_\bullet) \otimes N \rightarrow H_n(C_\bullet \otimes N)$  is natural)*

$$0 \longrightarrow H_n(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \text{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

*Proof.*

We have the short exact sequence of chain complexes of free  $R$ -modules

$$0 \longrightarrow Z_\bullet \hookrightarrow C_\bullet \longrightarrow B_{\bullet-1} \twoheadrightarrow 0$$

where  $Z_n = \ker(\partial : C_n \rightarrow C_{n-1})$  and  $B_n = \text{im}(\partial : C_{n+1} \rightarrow C_n)$  and the boundary maps on  $Z_\bullet$  and  $B_{\bullet-1}$  are zeros. Note that,  $B_{n-1}$  is free since it is a submodule of free  $R$ -module  $C_n$ , then the sequence splits. Then, the sequence below is exact and splits

$$0 \longrightarrow Z_\bullet \otimes N \hookrightarrow C_\bullet \otimes N \twoheadrightarrow B_{\bullet-1} \otimes N \longrightarrow 0$$

where the boundary maps on  $Z_\bullet \otimes N$  and  $B_{\bullet-1} \otimes N$  are zeros and boundary map on  $C_\bullet \otimes N$  is  $\partial \otimes 1 : c \otimes n \mapsto \partial c \otimes n$ . That induces a long exact sequence

$$\begin{array}{ccccccc}
& & & & \dots & \longrightarrow & H_{n+1}(B_{\bullet-1} \otimes N) \\
& & & & & \swarrow & (i_n)_* \\
H_n(Z_\bullet \otimes N) & \longrightarrow & H_n(C_\bullet \otimes N) & \longrightarrow & H_n(B_{\bullet-1} \otimes N) \\
& & & \swarrow & (i_{n-1})_* \\
H_{n-1}(Z_\bullet \otimes N) & \longrightarrow & \dots & & & & 
\end{array}$$

where the connecting homomorphism  $(i_n)_*$  is induced by inclusion map  $i_n : B_{n-1} \otimes N \rightarrow Z_n \otimes N$

$$\begin{array}{ccc}
& & B_{n-1} \otimes N \\
& \swarrow & \nearrow i_n \\
Z_n \otimes N & \longleftarrow & C_n \otimes N
\end{array}$$

That induces a short exact sequence

$$0 \longrightarrow \text{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

We have another short exact sequence

$$0 \longrightarrow B_n \xrightarrow{j} Z_n \twoheadrightarrow H_n(C_\bullet) \longrightarrow 0$$

Then, the sequence below is exact (*see the motivation of left derived functor*)

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \text{Tor}_1(Z_n, N) & \longrightarrow & \text{Tor}_1(H_n(C_\bullet), N) \\
& & & \swarrow & \\
B_n \otimes N & \xrightarrow{j \otimes 1 = i_n} & Z_n \otimes N & \twoheadrightarrow & H_n(C_\bullet) \otimes N & \longrightarrow & 0
\end{array}$$

Note that, both  $i_n$  and  $(i_n)_*$  are induced from the inclusion  $B_n \rightarrow Z_n$  and we have a natural transformation of functors from the category of pairs of a chain complex and an  $R$ -module to  $\text{Ab}$ ,

$$\begin{aligned}
(B_\bullet, N) &\mapsto H_n(B_\bullet) \otimes N \\
(B_\bullet, N) &\mapsto H_n(B_\bullet \otimes N)
\end{aligned}$$

so  $(i_n)_* = i_n$

$$B_n \hookrightarrow Z_n$$

$$\begin{array}{ccc} B_n \otimes N = H_n(B_\bullet) \otimes N & \xrightarrow{j \otimes 1 = i_n} & Z_n \otimes N = H_n(Z_\bullet) \otimes N \\ \cong \downarrow & & \downarrow \cong \\ H_n(B_\bullet \otimes N) & \xrightarrow{(i_n)_*} & H_n(Z_\bullet \otimes N) \end{array}$$

As  $Z_n$  is free,  $\text{Tor}_1(Z_n, N) = 0$ , then

$$\begin{aligned} \text{coker}(i_n)_* &= H_n(C_\bullet) \otimes N \\ \text{ker}(i_n)_* &= \text{Tor}_1(H_n(C_\bullet), N) \end{aligned}$$

We have the short exact sequence

$$0 \longrightarrow H_p(C_\bullet) \otimes N \longrightarrow H_n(C_\bullet \otimes N) \longrightarrow \text{Tor}_1(H_{n-1}(C_\bullet), N) \longrightarrow 0$$

The split of this sequence is from the map  $\beta : H_n(C_\bullet \otimes N) \rightarrow \text{coker}(i_n)_*$  induced from projection map  $C_n \rightarrow Z_n$   $\square$

### 3 KÜNNETH THEOREM FOR CHAIN COMPLEXES OF $R$ -MODULES

**Theorem 3** (Künneth theorem). *In the category  $R\text{-Mod}$  where  $R$  is a PID, let  $C_\bullet, D_\bullet$  be chain complexes of  $R$ -modules, and  $C_\bullet$  is degree-wise free (each  $C_n$  is a free  $R$ -module). Then, there is a natural short exact sequence (homology cross product is natural)*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

and this sequence splits (but not naturally)

*Proof.*

1. **Case 1:** boundary map of  $C_\bullet$  is zero

The boundary map  $(C_\bullet \otimes D_\bullet)_n \rightarrow (C_\bullet \otimes D_\bullet)_{n-1}$  is the linear extension of

$$\begin{aligned} \partial : C_p \otimes D_q &\rightarrow C_p \otimes D_{q-1} \\ c \otimes d &\mapsto (-1)^{|c|} c \otimes \partial d \end{aligned}$$

Hence, we can write the tensor product  $C_\bullet \otimes D_\bullet$  as a direct sum of chain complexes

$$C_\bullet \otimes D_\bullet = \bigoplus_p C_p \otimes D_{\bullet-p}$$

We have

$$\begin{aligned} H_n(C_\bullet \otimes D_\bullet) &= H_n\left(\bigoplus_p C_p \otimes D_{\bullet-p}\right) \\ &= \bigoplus_p H_n(C_p \otimes D_{\bullet-p}) \\ &= \bigoplus_p C_p \otimes H_n(D_{\bullet-p}) && (C_p \text{ is free, cons of UCT}) \\ &= \bigoplus_{p+q=n} C_p \otimes H_q(D_\bullet) && (\text{shifted chain complex}) \end{aligned}$$

2. **Case 2:**  $C_\bullet$  is an arbitrary chain complex

We have the short exact sequence of chain complexes of free  $R$ -modules

$$0 \longrightarrow Z_\bullet \hookrightarrow C_\bullet \longrightarrow B_{\bullet-1} \twoheadrightarrow 0$$

where  $Z_n = \ker(\partial : C_n \rightarrow C_{n-1})$  and  $B_n = \text{im}(\partial : C_{n+1} \rightarrow C_n)$  and the boundary maps on  $Z_\bullet$  and  $B_{\bullet-1}$  are zeros. Note that,  $B_{n-1}$  is free since it is a submodule of free  $R$ -module  $C_n$ , then the sequence splits. Then, the sequence below is exact and splits (*different from the proof of UCT, this uses split, direct sum, tensor product of sequence of chain complexes*)

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \hookrightarrow C_\bullet \otimes D_\bullet \twoheadrightarrow B_{\bullet-1} \otimes D_\bullet \longrightarrow 0$$

That induces a long exact sequence in homology

$$\begin{array}{ccccccc} & & & & \dots & \longrightarrow & H_{n+1}(B_{\bullet-1} \otimes D_\bullet) \\ & & & & & \swarrow & (i_n)_* \\ H_n(Z_\bullet \otimes D_\bullet) & \longleftarrow & H_n(C_\bullet \otimes D_\bullet) & \longrightarrow & H_n(B_{\bullet-1} \otimes D_\bullet) & & \\ & & & & & \swarrow & (i_{n-1})_* \\ H_{n-1}(Z_\bullet \otimes D_\bullet) & \longleftarrow & \dots & & & & \end{array}$$

where the connecting homomorphism  $(i_n)_*$  are induced by inclusion map  $i_n : (B_{\bullet-1} \otimes D_\bullet)_{n+1} \rightarrow (Z_\bullet \otimes D_\bullet)_n$

$$\begin{array}{ccc} & & (B_{\bullet-1} \otimes D_\bullet)_{n+1} \\ & & \swarrow \\ (Z_\bullet \otimes D_\bullet)_n & \longleftarrow & (C_\bullet \otimes D_\bullet)_n \\ & & \swarrow \\ & & (B_{\bullet-1} \otimes D_\bullet)_{n+1} \end{array}$$

That induces a short exact sequence

$$0 \longrightarrow \text{coker}(i_n)_* \longrightarrow H_n(C_\bullet \otimes D_\bullet) \longrightarrow \ker(i_{n-1})_* \longrightarrow 0$$

We have another short exact sequence

$$0 \longrightarrow B_p \xhookrightarrow{j} Z_p \twoheadrightarrow H_p(C_\bullet) \longrightarrow 0$$

Then, the sequence below is exact (*see the motivation of left derived functor*)

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Tor}_1(Z_p, H_q(D_\bullet)) & \longrightarrow & \text{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) & & \\ & & & & & \swarrow & \\ B_p \otimes H_q(D_\bullet) & \xleftarrow{j \otimes 1 = i_n} & Z_p \otimes H_q(D_\bullet) & \twoheadrightarrow & H_p(C_\bullet) \otimes H_q(D_\bullet) & \longrightarrow & 0 \end{array}$$

Take the direct sum over all pairs  $p+q = n$  and note that both  $i_n$  and  $(i_n)_*$  are induced from the the inclusion  $B_p \rightarrow Z_p$  we have a natural transformation of functors from the category of pairs of chain complexes to Ab

$$\begin{aligned} (A_\bullet, B_\bullet) &\mapsto H_n(A_\bullet \otimes B_\bullet) \\ (A_\bullet, B_\bullet) &\mapsto \bigoplus_{p+q=n} H_p(A_\bullet) \otimes H_q(B_\bullet) \end{aligned}$$

so  $(i_n)_* = i_n$

$$\begin{array}{ccc} B_p & \hookrightarrow & Z_p \\ \bigoplus_{p+q=n} B_p \otimes H_q(D_\bullet) & \xleftarrow{j \otimes 1 = i_n} & \bigoplus_{p+q=n} Z_p \otimes H_q(D_\bullet) \\ \downarrow = & & \downarrow = \\ H_{n+1}(B_{\bullet-1} \otimes D_\bullet) & \xrightarrow{(i_n)_*} & H_n(Z_\bullet \otimes D_\bullet) \end{array}$$

As  $Z_p$  is free,  $\text{Tor}_1(Z_p, H_q(D_\bullet)) = 0$ , then

$$\begin{aligned}\text{coker}(i_n)_* &= \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \\ \text{ker}(i_n)_* &= \bigoplus_{p+q=n} \text{Tor}_1(H_p(C_\bullet), H_q(D_\bullet))\end{aligned}$$

We have the short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

The split of this sequence is from the map  $\beta : H_n(C_\bullet \otimes D_\bullet) \rightarrow \text{coker}(i_n)_*$  induced from projection map  $(C_\bullet \otimes D_\bullet)_n \rightarrow (Z_\bullet \otimes D_\bullet)_n$  □

## 4 REAL-WORLD APPLICATIONS IN ALGEBRAIC TOPOLOGY

### 4.1 UNIVERSAL COEFFICIENT THEOREM FOR TOPOLOGICAL SPACES

**Theorem 4** (universal coefficient theorem - UCT). *Let  $X$  be a topological space and  $C_\bullet(X)$  be the singular chain complex of  $X$ . The singular chain complex with coefficients  $N$  be defined by*

$$C_\bullet(X, N) = C_\bullet(X) \otimes N$$

The homology group with coefficients  $N$  is defined by

$$H_n(X, N) = H_n(C_\bullet(X, N))$$

Then, there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes N \longrightarrow H_n(X, N) \longrightarrow \text{Tor}_1(H_{n-1}(X), N) \longrightarrow 0$$

and this sequence splits (but not naturally)

### 4.2 EILENBERG-ZILBER THEOREM

**Remark 9** ( $\text{Fun}(C, R\text{-Mod})$ ). *Given a category  $C$ ,  $\text{Fun}(C, R\text{-Mod})$  is a **pointed preadditive category with kernels** (more precisely, abelian category - will define in the future).*

*Given a morphism  $F \rightarrow G$  in  $\text{Fun}(C, R\text{-Mod})$  (a natural transformation from  $F$  to  $G$ ), then the kernel of  $F \rightarrow G$  is a morphism  $K \rightarrow F$  such that  $K(X)$  is the kernel of  $F(X) \rightarrow G(X)$  for all  $X \in \text{ob } C$*

**Remark 10** (models define projective class in  $\text{Fun}(C, R\text{-Mod})$ ). *Let  $\mathcal{M}$  be any set of objects in  $C$  (called models), then  $\mathcal{M}$  defines a projective class  $(\mathcal{P}, \mathcal{E})$  in  $\text{Fun}(C, R\text{-Mod})$  where a morphism  $G \rightarrow F$  is an epimorphism (relative to  $\mathcal{M}$ ) if for all  $M \in \mathcal{M}$ ,  $G(M) \rightarrow F(M)$  is surjective. Then, the following are equivalent*

1.  $P \in \text{ob } \text{Fun}(C, R\text{-Mod})$  is projective
2.  $P$  is a **retract of coproduct** of  $R\text{Hom}(M, -)$  for some  $M \in \mathcal{M}$  where  $\text{Hom}(M, -)$  is a functor  $R\text{-Mod} \rightarrow \text{Set}$ ,  $R$  is the free  $R$ -module functor  $\text{Set} \rightarrow R\text{-Mod}$ . In the case of  $R$ -module, **retract of coproduct** is the **direct summand** of a  $R$ -module

*Proof. **TODO - prove using Yoneda lemma*** □

**Remark 11.** *Let  $C = \text{Top}$  and model  $\mathcal{M} = \{\Delta^n : n = 0, 1, \dots\}$ , then for each  $n$ ,  $R\text{Hom}(\Delta^n, -)$  is projective. Note that,  $R\text{Hom}(\Delta^n, -)$  is the  $n$ -singular chain complexes with coefficients in  $R$  denoted by  $C_n$ . Moreover, the sequence below is a projective resolution of the zero-th homology functor  $H_0$*

$$0 \longleftarrow H_0 \xleftarrow{\epsilon} C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \dots$$

*Let  $\text{Ev}_X$  be the evaluation functor of topological space  $X$ , then the left derived functor  $\text{Ev}_X$  on  $H_0$  is the  $n$ -th singular homology of  $X$*

$$(L_n \text{Ev}_X)(H_0) = H_n(C_\bullet(X)) = H_n(X)$$

**Theorem 5** (Eilenberg-Zilber theorem). *Let  $X, Y$  be topological spaces and  $C_\bullet$  be the singular chain complex functor from  $\text{Top}$  to  $\text{R-Mod}$ , then there are two chain maps*

$$\begin{aligned} F : C_\bullet(X \times Y) &\rightarrow C_\bullet(X) \otimes C_\bullet(Y) \\ G : C_\bullet(X) \otimes C_\bullet(Y) &\rightarrow C_\bullet(X \times Y) \end{aligned}$$

*such that  $FG$  and  $GF$  are chain homotopic to identity. That is,  $C_\bullet(X \times Y)$  and  $C_\bullet(X) \otimes C_\bullet(Y)$  are of the same chain homotopy type.*

*Proof.*

*idea of proof*

In the category of  $\text{Top} \times \text{Top}$ , let model  $\mathcal{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$ . These functors  $\text{Top} \times \text{Top} \rightarrow \text{R-Mod}$  are projective

$$\begin{aligned} C_n(X \times Y) &= R[\text{Hom}(\Delta^n, X) \times \text{Hom}(\Delta^n, Y)] \\ C_p(X) \otimes C_q(Y) &= R[\text{Hom}(\Delta^p, X) \times \text{Hom}(\Delta^q, Y)] \end{aligned}$$

As  $H_0(X \times Y) \rightarrow H_0(X) \rightarrow H_0(Y)$  is an isomorphism, by FTHA, that induces a chain homotopy equivalence.

$$\begin{array}{ccccccc} 0 & \longleftarrow & H_0(X \times Y) & \longleftarrow & C_0(X \times Y) & \longleftarrow & C_1(X \times Y) \longleftarrow \dots \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longleftarrow & H_0(X) \otimes H_0(Y) & \longleftarrow & (C_\bullet(X) \otimes C_\bullet(Y))_0 & \longleftarrow & (C_\bullet(X) \otimes C_\bullet(Y))_1 \longleftarrow \dots \end{array}$$

□

**Corollary 2.** *Same chain homotopy type induces isomorphism in homology, that is*

$$H_n(X \times Y) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y))$$

### 4.3 KÜNNETH THEOREM FOR TOPOLOGICAL SPACES

**Theorem 6** (Künneth theorem). *In the category  $\text{R-Mod}$  where  $R$  is a PID, let  $C_\bullet, D_\bullet$  be chain complexes of  $R$ -modules, and  $C_\bullet$  is degree-wise free (each  $C_n$  is a free  $R$ -module). Then, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \xrightarrow{\times} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \longrightarrow 0$$

*and this sequence splits (but not naturally)*

**Theorem 7** (Künneth theorem). *Let  $X, Y$  be topological spaces and  $R$  be a PID, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(X; R), H_q(Y; R)) \longrightarrow 0$$

*and this sequence splits (but not naturally).*

**Corollary 3.** *If  $H_\bullet(X; R)$  is torsion free over  $R$  (of the form  $R \oplus R \oplus \dots$  without any  $R/nR$  term), then*

$$H_n(X \times Y) = \bigoplus_{p+q=n} H_p(X; R) \otimes H_q(Y; R)$$