ma5210 assignment 1

Nguyen Ngoc Khanh - A0275047B

August 2024

Let $(M, \mathcal{T})$ be a topological space. We assume that $(M, \mathcal{T})$ satisfies the following conditions:

1. $(M, \mathcal{T})$ is Hausdorff and second countable
2. $\{U_1, U_2, U_3\}$ is an open cover of M
3. For each $j = 1, 2, 3$, there is a continuous function $h_j : U_j \to \mathbb{C}^n$

1 Question 1

Suppose we want $(M, \mathcal{T})$ to be a topological manifold with an atlas

$$\mathcal{C} = \{(U_1, h_1), (U_2, h_2), (U_3, h_3)\}$$

what additional property/properties do we need to impose on (h_j, U_j)'s

Hint: you have to change to codomain of each h_j

Answer. For $\mathcal{C}$ to be an atlas, we require each h_j to be a homeomorphism from U_j to an open subset of $\mathbb{C}^n$. Two properties for each j

1. $h_j : U_j \to U'_j$ where U'_j is an open set in $\mathbb{C}^n$
2. h_j is a homeomorphism

□

2 Question 2

Suppose $(M, \mathcal{T})$ is a topological manifold with atlas $\mathcal{C}$ as in Question 1.

What additional property/properties do we need to impose on $\mathcal{C}$ so that $(M, \mathcal{T})$ is a complex analytic manifold with atlas $\mathcal{C}$

Answer. For each i, j such that $U := U_i \cap U_j \neq \varnothing$, the transition function

$$t_{ij} = h_j h_i^{-1} : h_i(U) \to h_j(U)$$

is complex analytic. □

3 Question 3

Suppose $(M, \mathcal{T})$ is a complex analytic manifold with atlas $\mathcal{C}$ as in Question 2.

Let U_4 be an open subset of U_1 and let h_4 be the restriction of h_1 on U_4. Let $U'_4 = \operatorname{im} h_4 \subseteq \mathbb{C}^n$. Is

$$\mathcal{D} = \{(U_1, h_1), (U_2, h_2), (U_3, h_3), (U_4, h_4)\}$$

an atlas of $(M, \mathcal{T})$ as a complex analytic manifold?

Answer. $\mathcal{D}$ is an atlas of $(M, \mathcal{T})$ as a complex analytic manifold □

Proof. We will verify that (1) $h_4 : U_4 \to U'_4$ is a homeomorphism from open set $U_4 \subseteq M$ to open set $U'_4 \subseteq \mathbb{C}^n$ and (2) h_{i4} and h_{4i} are complex holomorphic for all i such that $U_i \cap U_4 \neq \varnothing$

(1) h_4 is a restriction of homeomorphism h_1, so h_4 is still a homeomorphism. Since U_4 open in U_1 and U_1 open in M, U_4 is also open in M, therefore, U'_4 is open set in $\mathbb{C}^n$ as it is the image of open set U_4 under open mapping h_4. Hence, $h_4 : U_4 \to U'_4$ is a homeomorphism from open set $U_4 \subseteq M$ to open set $U'_4 \subseteq \mathbb{C}^n$

(2) Let $U_i \cap U_4 \neq \varnothing$, then $U_i \cap U_1 \neq \varnothing$, the transition functions h_{1i} and h_{i1} are complex holomorphic. As the transition h_{4i} is restriction of h_{1i} on open set $h_4(U_4 \cap U_i)$, the transition h_{i4} is restriction of h_{i1} on open set $h_i(U_4 \cap U_i)$, they are complex holomorphic

□

4 Question 4

Question 3 provides a method of adding more charts to the atlas $\mathcal{C}$ by restricting the h_j's to open subsets. Could you think of other method(s) of creating more charts to add to the atlas $\mathcal{C}$?

Answer. Some methods to creating more charts

1. Let $h_i : U_i \to U'_i \subseteq \mathbb{C}^n$ be a chart and $g : U'_i \to U''_i \subseteq \mathbb{C}^n$ be a homeomorphism such that for every chart $h_j : U_j \to U'_j$ with $U := U_j \cap U_i \neq \varnothing$, the composition $gt_{ji} = gh_ih_j^{-1} : h_j(U) \to gh_i(U)$ and $t_{ij}g^{-1} = h_jh_i^{-1}g^{-1} : gh_i(U) \to h_j(U)$ are still complex holomorphic.

$$\begin{tikzcd}
\mathbb{C}^n & M & \mathbb{C}^n & \mathbb{C}^n \\
U'_j \arrow[u, hook] & U_i \cap U_j \arrow[u, hook] \arrow[l, "h_j"'] \arrow[r, "h_i"] \arrow[rr, bend right, "gh_i"'] & U'_i \arrow[u, hook] \arrow[r, "g"] & U''_i \arrow[u, hook]
\end{tikzcd}$$

One example of compatible g is the translation, let any $p \in U_i$, define $g(z) = z - h_i(p)$ so that the composition gh_i maps p to $0 \in \mathbb{C}^n$ (this is used in the proof of "global holomorphic functions are necessarily constant")

Once we can find g, define a new chart by
$$(U_i, gh_i : U_i \to U''_i)$$

Let $i = j$, we see that g is necessarily holomorphic. Furthermore, composition of n-variables holomorphic functions are n-variables holomorphic, we conclude that g is holomorphic homeomorphism [1].

Proving composition of n-variables holomorphic functions are n-variables holomorphic. Let $y : A \to \mathbb{C}^n$, $x : y(A) \to \mathbb{C}^n$ be complex holomorphic. We will show that $xy : A \to \mathbb{C}^n$ is holomorphic.

For each $i = 1, ..., n$, y_i is holomorphic, for every $z^{(0)} = (z_1^{(0)}, ..., z_n^{(0)}) \in A$, there exists an open ball B_i^z centered at $z^{(0)}$ of radius $r_i^z > 0$ such that for all $z \in B_i^z$, for each $i = 1, ..., n$, the series below is absolutely convergent

$$y_i(z) = \sum_{(a_1,...,a_n) \in \mathbb{N}^n} c_{a_1,...,a_n}(z_1 - z_1^{(0)})^{a_1} ... (z_n - z_n^{(0)})^{a_n}$$

Similarly, for each $j = 1, ..., n$, z_j is holomorphic, there exists an open ball B_j^y centered at $y(z^{(0)})$ of radius $r_j^y > 0$, for every $z \in y^{-1}B_j^y \cap B_1^z \cap ... \cap B_n^z$, the series below is absolutely convergent

$$x_jy(z) = \sum_{(b_1,...,b_n) \in \mathbb{N}^n} d_{b_1,...,b_n}(y_1(z) - y_1(z^{(0)}))^{b_1} ... (y_n(z) - y_n(z^{(0)}))^{b_n}$$

By absolute convergence, rearrange the terms, $x_jy(z)$ is a convergent power series when $z \in y^{-1}B_j^y \cap B_1^z \cap ... \cap B_n^z$. We choose an open ball B_j centered at $z^{(0)}$, hence each x_jy is holomorphic. Therefore, xy is n-variables holomorphic.

□

[1] I am still not sure that n-variables holomorphic is necessarily homeomorphism

5 Question 5

Suppose $(M, \mathcal{T})$ is a complex analytic manifold with atlas $\mathcal{C}$ as in Question 2. Show that $(M, \mathcal{T})$ is a real smooth manifold of (real) dimension $2n$.

Proof. Let $g : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the canonical homeomorphism from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ defined by

$$g : (a_1 + ib_1, ..., a_n + ib_n) \mapsto (a_1, b_1, ..., a_n, b_n)$$

Let $\mathcal{C} = \{(U_i, h_i)\}_{i \in I}$, define

$$\mathcal{D} = \{(U_i, gh_i)\}_{i \in I}$$

We will prove that $\mathcal{D}$ makes $(M, \mathcal{T})$ a real analytic manifold of dimension $2n$ by verifying transition function being real analytic.

Let $U := U_i \cap U_j \neq \varnothing$, the new transition function is $gh_jh_i^{-1}g^{-1} : gh_i(U) \to gh_j(U)$ where $h_jh_i^{-1}$ is complex holomorphic. Let $z^{(0)} = (a_1^{(0)} + ib_1^{(0)}, ..., a_n^{(0)} + ib_n^{(0)}) \in h_i(U)$, there is an open ball B centered at $z^{(0)}$ of radius $r > 0$ such that for all $z = (a_1 + ib_1, ..., a_n + ib_n) \in B$, the series below is absolutely convergent

$$\begin{aligned} h_jh_i^{-1}(z) &= \sum_{(d_1,...,d_n) \in \mathbb{N}^n} c_{d_1,...,d_n} (z_1 - z_1^{(0)})^{d_1} ... (z_n - z_n^{(0)})^{d_n} \\ &= \sum_{(d_1,...,d_n) \in \mathbb{N}^n} c_{d_1,...,d_n} [(a_1 - a_1^{(0)}) + i(b_1 - b_1^{(0)})]^{d_1} ... [(a_n - a_n^{(0)}) + i(b_n - b_n^{(0)})]^{d_n} \end{aligned}$$

By absolute convergence, rearrange the terms, let $x^{(0)} = (a_1^{(0)}, b_1^{(0)}, ..., a_n^{(0)}, b_n^{(0)})$ and $x = (a_1, b_1, ..., a_n, b_n)$, $gh_jh_i^{-1}g^{-1}(x)$ can be written as an absolutely convergent power series. Hence, for all $x^{(0)} \in gh_i(U)$, there exists an open ball B' centered at $z^{(0)}$ of radius $r > 0$ such that for all $x \in B'$, $gh_jh_i^{-1}g^{-1}$ can be written as an absolutely convergent power series. That is, $gh_jh_i^{-1}g^{-1}$ is analytic.

□