ma5210 assignment 2

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August 2024

1 Questions 1-4

1.1 Question 1

Let A be an m by m real matrix. Suppose $A^2 = -I_m$ where I_m is the identity matrix

1. Show that A is an invertible matrix
2. Show that m is an even integer

Answer. The inverse of A is $A^{-1} = -A$. If $A^2 = -I_m$, then

$$\det(A)^2 = \det(A^2) = \det(-I_m) = (-1)^m$$

If m is odd, then we have $\det(A)^2 = -1$ which is a contradiction since $\det(A)$ is real. □

1.2 Question 2

Let V be a $\mathbb{R}$-vector space of finite dimension m. Let J be a complex structure on V i.e. $J : V \to V$ is an $\mathbb{R}$-linear transformation such that $J^2 = -\operatorname{id}_V$

1. Show that J is an invertible linear transformation
2. Show that m is an even integer

Answer. The inverse of J is $J^{-1} = -J$. Let $B = \{e_1, e_2, ..., e_m\}$ be a basis of V. Then, let $A \in \mathbb{R}^{m\times m}$ be the matrix of J over B. Then we have, $A^2 = -I_m$. Therefore, m is even. □

1.3 Question 3

Let U' be an open subset of $\mathbb{R}^6$. Let ϕ be a section in $\mathcal{E}^2(U') = \mathcal{E}(U', \wedge^2 T^*(\mathbb{R}^6))$. We write

$$\phi(x) = \sum_{1\le j<k\le 6} f_{jk}(x) dx_j \wedge dx_k$$

where $x = (x_1, x_2, ..., x_6) \subseteq U'$ and $f_{jk} : U' \to \mathbb{R}$ is a smooth function.

1. Compute $d\phi \subseteq \mathcal{E}^3(U')$ in terms of $dx_1, dx_2, ..., dx_6$
2. Show that $d^2\phi = 0$ in $\mathcal{E}^4(U')$

1.3.1 Compute $d\phi \subseteq \mathcal{E}^3(U')$

$$\begin{aligned} d\phi &= d\left(\sum_{1\le j<k\le 6} f_{jk} dx_j \wedge dx_k\right) \\ &= \sum_{1\le j<k\le 6} df_{jk} \wedge dx_j \wedge dx_k \\ &= \sum_{1\le j<k\le 6} \left(\sum_{i=1}^{6} \frac{\partial f_{jk}}{\partial x_i} dx_i\right) \wedge dx_j \wedge dx_k \\ &= \sum_{1\le j<k\le 6} \sum_{i=1}^{6} \frac{\partial f_{jk}}{\partial x_i} dx_i \wedge dx_j \wedge dx_k \end{aligned}$$

1.3.2 Show that $d^2\phi = 0$ in $\mathcal{E}^4(U')$

We will show that $d^2 = 0$ for the general case.
Let $U' \subseteq \mathbb{R}^n$ be an open set, let $0 \leq m < n$, let ϕ be a section in $\mathcal{E}^m(U') = \mathcal{E}(U', \wedge^m T^*(\mathbb{R}^n))$. Let

$$[n] = \{1, ..., n\}$$

for any subset $\sigma \subseteq [n]$ of size m where $\sigma = \{i_1 < i_2 < ... < i_m\}$, let

$$dx_\sigma = dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_m} \in \mathcal{E}^m(U')$$

Then a basis of $\mathcal{E}^m(U')$ is

$$D_m = \{dx_\sigma : \sigma \subseteq [n], |\sigma| = m\}$$

We can write $\phi \in \mathcal{E}^m(U')$ as

$$\phi = \sum_{\sigma \in D_m} f_\sigma dx_\sigma$$

By defintion of $d : \mathcal{E}^m(U') \to \mathcal{E}^{m+1}(U')$, we have

$$d\phi = \sum_{\sigma \in D_m} df_\sigma \wedge dx_\sigma = \sum_{\sigma \in D_m} \sum_{i=1}^{n} \frac{\partial f_\sigma}{\partial x_i} dx_i \wedge dx_\sigma$$

And

$$\begin{aligned} d^2\phi &= \sum_{\sigma \in D_m} \sum_{i=1}^{n} d\left(\frac{\partial f_\sigma}{\partial x_i}\right) \wedge dx_i \wedge dx_\sigma \\ &= \sum_{\sigma \in D_m} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_\sigma}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_\sigma \end{aligned}$$

For all pairs of $(i, j) \in [n] \times [n]$, if $i = j$, then $dx_i \wedge dx_i \wedge dx_\sigma = 0$. If $i \neq j$, we have

$$\frac{\partial^2 f_\sigma}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_\sigma + \frac{\partial^2 f_\sigma}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_\sigma = 0$$

Therefore, $d^2 = 0$. This is also true for the case $m < 0$ or $n \leq m$ since $d^2 : \mathcal{E}^m(U') \to \mathcal{E}^{m+2}(U')$ is a linear map from or to a zero dimensional vector space.

1.4 Question 4

Let M be a real smooth $\mathcal{E}$-manifold of dimension 6.

1. Let $h : U \to U'$ be a chart where U' is an open subset of $\mathbb{R}^6$. Let $\omega \in \mathcal{E}^2(U) = \mathcal{E}(U, \wedge^2 T^*(M))$. Show that $d^2\omega = 0$
2. Consider the composition of maps

$$\begin{tikzcd} \mathcal{E}^2(M) \arrow[r, "d"] \arrow[rr, bend right, "d^2"'] & \mathcal{E}^3(M) \arrow[r, "d"] & \mathcal{E}^4(M) \end{tikzcd}$$

True or false: $d^2 = 0$, give proof, counterexample

1.4.1 Show that $d^2\omega = 0$

The diagram below commutes ($m = 2$)

$$\begin{tikzcd} \mathcal{E}^m(U') \arrow[r, "d"'] \arrow[rr, bend left, "d^2=0"] \arrow[d, "\cong"] & \mathcal{E}^{m+1}(U') \arrow[r, "d"'] \arrow[d, "\cong"] & \mathcal{E}^{m+2}(U') \arrow[d, "\cong"] \\ \mathcal{E}^m(U) \arrow[r, dotted, "d"] \arrow[rr, dotted, bend right, "d^2"'] & \mathcal{E}^{p+1}(U) \arrow[r, dotted, "d"] & \mathcal{E}^{m+2}(U) \end{tikzcd}$$

Therefore, $d^2 : \mathcal{E}^m(U) \to \mathcal{E}^{m+2}(U)$ is a composition of

$$\begin{aligned}\cong &: \mathcal{E}^m(U) \to \mathcal{E}^m(U') \\ d^2 &: \mathcal{E}^m(U') \to \mathcal{E}^{m+2}(U') \\ \cong &: \mathcal{E}^{m+2}(U') \to \mathcal{E}^{m+2}(U)\end{aligned}$$

$d^2 : \mathcal{E}^m(U') \to \mathcal{E}^{m+2}(U')$ is a zero map implies $d^2 : \mathcal{E}^m(U) \to \mathcal{E}^{m+2}(U)$ is a zero map

1.4.2 True or false: $d^2 = 0$, give proof, counterexample

Let $U \subseteq M$ be a chart of M. The diagram below commutes

$$\begin{tikzcd}
\mathcal{E}^m(U) \arrow[r, "d"'] \arrow[rr, bend left, "d^2=0"] & \mathcal{E}^{m+1}(U) \arrow[r, "d"'] & \mathcal{E}^{m+2}(U) \\
\mathcal{E}^m(M) \arrow[u, twoheadrightarrow, "r_U^M"] \arrow[r, dashed, "d"] \arrow[rr, dashed, bend right, "d^2"'] & \mathcal{E}^{m+1}(M) \arrow[u, twoheadrightarrow, "r_U^M"] \arrow[r, dashed, "d"] & \mathcal{E}^{m+2}(M) \arrow[u, twoheadrightarrow, "r_U^M"]
\end{tikzcd}$$

Let $\omega \in \mathcal{E}^m(M) = \mathcal{E}(M, \wedge^m T^*(M))$, then the restriction of $d^2\omega$ on U denoted by $(d^2\omega)|_U \in \mathcal{E}^{m+2}(U) = \mathcal{E}(U, \wedge^{m+2} T^*(M))$ is

$$(d^2\omega)|_U = r_U^M d^2\omega = d^2 r_U^M \omega = 0$$

$d^2\omega$ restricted to any chart is zero, therefore, it is zero globally.

2 Questions 5-10

In the last few questions, we clarify some connections between the real tangent spaces and complex tangent spaces of a complex manifold.

Let $M = \mathbb{C}^3$ which is a complex analytic manifold. When we consider M as a real manifold, we will denote it by M_0 to avoid confusion. We have $M_0 = \mathbb{R}^6$ and the bijection $\Phi : M \to M_0$ is given by

$$\Phi(z_1, z_2, z_3) = (x_1, y_1, x_2, y_2, x_3, y_3)$$

where $z_j = x_j + \sqrt{-1}y_j$ for $j = 1, 2, 3$.

We fix a point $x = (a_1, b_1, a_2, b_2, a_3, c_3) \in M_0$.

The real tangent space of M_0 at x is

$$T_x(M_0) = \mathbb{R} - \text{span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_3}\right\}$$

We set

$$\begin{aligned}\frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right) \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right)\end{aligned}$$

for $j = 1, 2, 3$

2.1 Question 5

Show that $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$ where

$$\begin{aligned}T^{1,0} &= \mathbb{C} - \text{span}\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\right\} \\ T^{0,1} &= \mathbb{C} - \text{span}\left\{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \frac{\partial}{\partial \bar{z}_3}\right\}\end{aligned}$$

Proof. A basis of $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C}$ is

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_3} \right\} \otimes_{\mathbb{R}} 1$$

Without confusion, we denote the basis vectors of $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\frac{\partial}{\partial x_j} := \frac{\partial}{\partial x_j} \otimes_{\mathbb{R}} 1 \text{ and } \frac{\partial}{\partial y_j} := \frac{\partial}{\partial y_j} \otimes_{\mathbb{R}} 1$$

It is clear that basis vectors of $T^{1,0} \oplus T^{0,1}$ are linear combinations of basis vectors of $T_x(M_0) \otimes \mathbb{C}$, that is

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \end{aligned}$$

Therefore, $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} \supseteq T^{1,0} \oplus T^{0,1}$. Moreover, basis vectors of $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C}$ are also linear combinations of basis vectors of $T^{1,0} \oplus T^{0,1}$

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{\partial}{\partial \bar{z}_j} + \frac{\partial}{\partial z_j} \\ \frac{\partial}{\partial y_j} &= i\left(\frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial z_j} \right) \end{aligned}$$

TODO - mistake here

Therefore, $T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} \subseteq T^{1,0} \oplus T^{0,1}$, then

$$T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$$

□

2.2 Question 6

The same point x is the point $(a_1 + \sqrt{-1}b_1, a_2 + \sqrt{-1}b_2, a_2 + \sqrt{-1}b_2, a_3 + \sqrt{-1}b_3) \in M$. The complex tangent space of M at x is

$$T_x(M) = T^{1,0} = \mathbb{C} - \text{span}\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3} \right\}$$

Indeed this is the space of derivations of the holomorphic germs at x.

We have shown in class that two tangent spaces $T_x(M)$ and $T_x(M_0)$ are isomorphic as a real vector spaces. Since $T_x(M)$ is a complex vector space, it gives a complex structure J on $T_x(M_0)$, i.e $J : T_x(M_0) \to T_x(M_0)$ is an $\mathbb{R}$-linear transformation satisfying $J^2 = -\,\text{id}$. Compute

$$J\left(\frac{\partial}{\partial x_j} \right) \text{ and } J\left(\frac{\partial}{\partial y_j} \right)$$

for $j = 1, 2, 3$

Proof. Let $t : T_x(M_0) \to T^{1,0} = T_x(M)$ be the isomorphism as real vector space.

$$\begin{tikzcd} T_x(M_0) \arrow[r, hook, "inc"] \arrow[dr, dashed, "t"'] & T_x(M_0) \otimes C \arrow[d, two heads, "proj"] \\ & T_{1,0} \end{tikzcd}$$

Then, $t\frac{\partial}{\partial x_j}$ and $t\frac{\partial}{\partial y_j}$ are

$$
\begin{tikzcd}
T_x(M_0) \arrow[r, hook, "inc"] \arrow[rr, bend left, "t"] & T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, two heads, "proj"] & T^{1,0} \\
\frac{\partial}{\partial x_j} \arrow[r, mapsto] & \frac{\partial}{\partial x_j} = \frac{\partial}{\partial \bar{z}_j} + \frac{\partial}{\partial z_j} \arrow[r, mapsto] & \frac{\partial}{\partial z_j} \\
\frac{\partial}{\partial y_j} \arrow[r, mapsto] & \frac{\partial}{\partial y_j} = i\left(\frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial z_j}\right) \arrow[r, mapsto] & -i\frac{\partial}{\partial z_j}
\end{tikzcd}
$$

$J : T_x(M_0) \to T_x(M_0)$ is defined by

$$
\begin{tikzcd}
T^{1,0} \arrow[r, "i"] & T^{1,0} \\
T_x(M_0) \arrow[u, "t"] \arrow[r, dashed, "J"'] & T_x(M_0) \arrow[u, "t"']
\end{tikzcd}
$$

Then, $J\frac{\partial}{\partial x_j}$ and $J\frac{\partial}{\partial y_j}$ are

$$
\begin{tikzcd}
T_x(M_0) \arrow[r, "t"'] \arrow[rrr, bend left, "J"] & T^{1,0} \arrow[r, "i"'] & T^{1,0} \arrow[r, "t^{-1}"'] & T_x(M_0) \\
\frac{\partial}{\partial x_j} \arrow[r, mapsto] & \frac{\partial}{\partial z_j} \arrow[r] & i\frac{\partial}{\partial z_j} \arrow[r] & -\frac{\partial}{\partial y_j} \\
\frac{\partial}{\partial y_j} \arrow[r, mapsto] & -i\frac{\partial}{\partial z_j} \arrow[r] & \frac{\partial}{\partial z_j} \arrow[r] & \frac{\partial}{\partial x_j}
\end{tikzcd}
$$

□

2.3 Question 7

The cotangent space at x is

$$T^*_x(M_0) = \mathbb{R} - \text{span}\{dx_1, dy_1, dx_2, dy_2, dx_3, dy_3\}$$

where $dx_j : T_x(M_0) \to \mathbb{R}$ is the $\mathbb{R}$-linear transformation such that

$$dx_j\left(\frac{\partial}{\partial x_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$dx_j\left(\frac{\partial}{\partial y_k}\right) = 0 \text{ for every } k$$

What is the defintion of dy_j for $j = 1, 2, 3$

Answer.

$$dy_j\left(\frac{\partial}{\partial y_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$dy_j\left(\frac{\partial}{\partial x_k}\right) = 0 \text{ for every } k$$

□

2.4 Question 8

The cotangent space of complex manifold M at x is

$$T_x^*(M) = \mathbb{C} - \text{span}\{dz_1, dz_2, dz_3\}$$

where $dz_j : T_x(M_0) \otimes_R \mathbb{C} \to \mathbb{C}$ is the $\mathbb{C}$-linear transformation such that

$$dz_j\left(\frac{\partial}{\partial z_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$
$$dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0 \text{ for every } k$$

What is the defintion of $d\bar{z}_j$ for $j = 1, 2, 3$

Answer.

$$d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$
$$d\bar{z}_j\left(\frac{\partial}{\partial z_k}\right) = 0 \text{ for every } k$$

□

2.5 Question 9

Show that

$$dz_j = dx_j + \sqrt{-1}dy_j$$
$$d\bar{z}_j = dx_j - \sqrt{-1}dy_j$$

for $j = 1, 2, 3$

Proof. We extend $dx_j : T_x(M_0) \to \mathbb{R}$ and $dy_j : T_x(M_0) \to \mathbb{R}$ into $dx_j : T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ and $dy_j : T_x(M_0) \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ canonically as follows

$$dx_j\left(\frac{\partial}{\partial x_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$
$$dx_j\left(\frac{\partial}{\partial y_k}\right) = 0 \text{ for every } k$$
$$dy_j\left(\frac{\partial}{\partial y_k}\right) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$
$$dy_j\left(\frac{\partial}{\partial x_k}\right) = 0 \text{ for every } k$$

We will verify that $dx_j + idy_j$ agrees with the defintion of dz_j, $dx_j - idy_j$ agrees with the defintion of $d\bar{z}_j$

$$\begin{aligned}(dx_j + idy_j)\frac{\partial}{\partial z_k} &= (dx_j + idy_j)\left(\frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)\right) \\ &= \frac{1}{2}\left(dx_j\frac{\partial}{\partial x_k} + dy_j\frac{\partial}{\partial y_k}\right) \\ &= \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}\end{aligned}$$

$$\begin{aligned}(dx_j + idy_j)\frac{\partial}{\partial \bar{z}_k} &= (dx_j + idy_j)\left(\frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)\right) \\ &= \frac{1}{2}\left(dx_j\frac{\partial}{\partial x_k} - dy_j\frac{\partial}{\partial y_k}\right) \\ &= 0\end{aligned}$$

$$\begin{aligned}(dx_j - idy_j)\frac{\partial}{\partial z_k} &= (dx_j - idy_j)\left(\frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)\right)\\ &= \frac{1}{2}\left(dx_j\frac{\partial}{\partial x_k} - dy_j\frac{\partial}{\partial y_k}\right)\\ &= 0\end{aligned}$$

$$\begin{aligned}(dx_j - idy_j)\frac{\partial}{\partial \bar{z}_k} &= (dx_j - idy_j)\left(\frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)\right)\\ &= \frac{1}{2}\left(dx_j\frac{\partial}{\partial x_k} + dy_j\frac{\partial}{\partial y_k}\right)\\ &= \begin{cases}1 & \text{if } k = j\\ 0 & \text{if } k \neq j\end{cases}\end{aligned}$$

□

2.6 Question 10

Let $f : M_0 \to \mathbb{C}$ be a smooth function. We warn that f does not have to be holomorphic function. We define

$$df = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}dx_j + \frac{\partial f}{\partial y_j}dy_j$$

which is a section in

$$\mathcal{E}^1(M_0, T(M)\otimes\mathbb{C}) = \mathcal{E}^1(M_0, T^{1,0}) \oplus \mathcal{E}^1(M_0, T^{0,1})$$

We define ∂f as the projection of df into $\mathcal{E}^1(M_0, T^{1,0})$. Show that

$$\partial f = \sum_{j=1}^{3} \frac{\partial f}{\partial z_j}dz_j$$

Proof. Note that

$$\begin{aligned}T(M)\otimes\mathbb{C} &= \coprod_{m\in M_0} \mathrm{Hom}(T_m(M_0)\otimes\mathbb{C}, \mathbb{C})\\ T^{1,0} &= \coprod_{m\in M_0} \mathrm{Hom}(T_m^{1,0}, \mathbb{C})\end{aligned}$$

where $T_m(M_0)$ is the tangent space at $m \in M$ of M_0 and $T_m(M_0)\otimes\mathbb{C} = T_m^{1,0}\oplus T_m^{0,1}$. The projection from $\mathcal{E}^1(M_0, T(M)\otimes\mathbb{C})$ into $\mathcal{E}^1(M_0, T^{1,0})$ is defined as follows

$$\begin{tikzcd} & T(M)\otimes\mathbb{C} = \coprod_{m\in M_0}\mathrm{Hom}(T_m(M_0)\otimes\mathbb{C}, \mathbb{C}) \arrow[d, two heads] \\ M_0 \arrow[ur, "\phi"] \arrow[r, dashed, "\psi"] & T^{1,0} = \coprod_{m\in M_0}\mathrm{Hom}(T_m^{1,0}, \mathbb{C}) \end{tikzcd}$$

$$\begin{aligned}\phi(m) &: T_m(M_0)\otimes\mathbb{C} \to \mathbb{C}\\ \psi(m) &: T_m^{1,0}\otimes\mathbb{C} \to \mathbb{C}\end{aligned}$$

$\phi \in \mathcal{E}^1(M_0, T(M)\otimes\mathbb{C})$ is projected into $\psi \in \mathcal{E}^1(M_0, T^{1,0})$ such that for all $m \in M_0$, $\psi(m) : T_m^{1,0}\otimes\mathbb{C} \to \mathbb{C}$ is a restriction of $\phi(m) : T_m(M_0)\otimes\mathbb{C} \to \mathbb{C}$.

Note that, $\mathcal{E}^1(M_0, T(M)\otimes\mathbb{C}), \mathcal{E}^1(M_0, T^{1,0}), \mathcal{E}^1(M_0, T^{0,1})$ are all $\mathcal{E}(M_0)$-algebra.

Note that $\frac{\partial f}{\partial x_j} : M_0 \to \mathbb{C}$ is defined by

$$\frac{\partial f}{\partial x_j} : m \mapsto \left.\frac{\partial f}{\partial x_j}\right|_m = \frac{\partial}{\partial x_j}[f]_m$$

where $[f]_m$ is the germ of f at $m \in M_0$. Similar for $\frac{\partial f}{\partial y_j}$. Therefore, from previous part, $\left.\frac{\partial}{\partial z_j}\right|_m = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\Big|_m$ implies

$$\left.\frac{\partial f}{\partial z_j}\right|_m = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)\Big|_m$$

Now, for any $m \in M_0$, we have

$$(df)(m) = \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_m dx_j|_m + \left.\frac{\partial f}{\partial y_j}\right|_m dy_j|_m$$

From previous part we have

$$\begin{aligned} dz_j|_m &= dx_j|_m + idy_j|_m \\ d\bar{z}_j|_m &= dx_j|_m - idy_j|_m \end{aligned}$$

Therefore,

$$\begin{aligned} dx_j|_m &= \frac{1}{2}(d\bar{z}_j|_m + dz_j|_m) \\ dy_j|_m &= \frac{i}{2}(d\bar{z}_j|_m - dz_j|_m) \end{aligned}$$

Then,

$$(df)(m) = \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_m \frac{1}{2}(d\bar{z}_j|_m + dz_j|_m) + \left.\frac{\partial f}{\partial y_j}\right|_m \frac{i}{2}(d\bar{z}_j|_m - dz_j|_m)$$

Restrict into $\mathrm{Hom}(T_m^{1,0}, \mathbb{C}) = T_m^*(M) = \mathbb{C} - \mathrm{span}\{dz_1|_m, dz_2|_m, dz_3|_m\}$, we have

$$\begin{aligned} (\partial f)(m) &= \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_m \frac{1}{2} dz_j|_m - \left.\frac{\partial f}{\partial y_j}\right|_m \frac{i}{2} dz_j|_m \\ &= \sum_{j=1}^{3} \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)\Big|_m dz_j|_m \\ &= \sum_{j=1}^{3} \left.\frac{\partial f}{\partial z_j}\right|_m dz_j|_m \end{aligned}$$

Then,

$$\partial f = \sum_{j=1}^{3} \frac{\partial f}{\partial z_j} dz_j$$

□