topbook_harder

Khanh Nguyen

June 2023

some harder or interesting problems in topbook2023.pdf

Common

I will put here some common definitions, propositions used in this notes.

Proposition 1 (Cover Proposition) *Given a set A, for all element $x \in A$ if $x \in U_x \subseteq A$ then*

$$\bigcup_{x \in A} U_x = A$$

Proof

We immediately have $\bigcup_{x \in A} U_x \subseteq A$. On the other hand, for all $x \in A$, $\{x\} \subseteq U_x$. Then

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x$$

Definition 1 (Indistinguishability and T_0-space) *Two points $a, b \in X$ are called indistinguishable if every open set in X either contains both a and b or contains none. A topological space is a T_0-space if all pairs of points are distinguishable.*

Definition 2 (T_1-space) *A topological space $(X, \mathcal{T})$ is said to be a T_1-space if every singleton set $\{x\}$ for $x \in X$ is closed in $(X, \mathcal{T})$*

Definition 3 (Hausdorff space or T_2-space) *A topological space $(X, \mathcal{T})$ is said to be a Hausdorff space or T_2-space if given any two distinct points $a, b \in X$, there exists two disjoint open sets U, V such that $a \in U$ and $b \in V$*

Definition 4 (Regular space) *A topological space $(X, \mathcal{T})$ is said to be a regular space if for any closed subset $A \subseteq X$ and any point $b \in X \setminus A$, there exists two disjoint open sets U, V such that $A \subseteq U$ and $b \in V$*

Definition 5 (T_3-space) *A topological space $(X, \mathcal{T})$ is said to be a T_3-space if it is a T_1-space and a regular space.*

Definition 6 (Initial segment topology)

$$(\mathbb{N}, \{\varnothing, \mathbb{N}\} \cup \{\{1, 2, ..., n\} : n \in \mathbb{N}\})$$

Definition 7 (Final segment topology)

$$(\mathbb{N}, \{\varnothing, \mathbb{N}\} \cup \{\{n, n+1, ...\} : n \in \mathbb{N}\})$$

Definition 8 (Coarser Topology and Finer Topology) *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. $\mathcal{T}_1$ is said to be a finer topology than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is a coarser topology than $\mathcal{T}_1$) if $\mathcal{T}_1 \supseteq \mathcal{T}_2$*

Definition 9 (Totally disconnected space) *A topological space $(X, \mathcal{T})$ is said to be a totally disconnected space if every non-empty connected subset is a singleton set.*

Definition 10 (Zero dimensional space) *A topological space $(X, \mathcal{T})$ is said to be a zero dimensional space if there is a basis for the topology consisting of clopen sets.*

Definition 11 (Local homeomorphism) *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be topological spaces. A map $f : X \to Y$ is said to be a local homeomorphism if each point $x \in X$ has an open neighbourhood U such that the restriction of f to U maps U homeomorphically into an open subspace V of $(Y, \mathcal{T}_1)$;*

that is, if the topology induced on an open neighbourhood U by $\mathcal{T}$ is $\mathcal{T}_2$ and topology induced on $V = f(U)$ by $\mathcal{T}_1$ is $\mathcal{T}_3$, then f is a homeomorphism of $(U, \mathcal{T}_2)$ onto $(Y, \mathcal{T}_1)$

Exercise 1.1.9

Exercise 2.3.4

Let $C[0,1]$ be the set of all continuous real-value functions on $[0,1]$

- Show that the collection $\mathcal{M} = \{M(f, \epsilon) : f \in C[0,1] \wedge \epsilon > 0\}$ where $M(f, \epsilon) = \{g : g \in C[0,1] \wedge \int_0^1 |f - g| < \epsilon\}$ is a basis for a topology $\mathcal{T}_1$ on $C[0,1]$
- Show that the collection $\mathcal{U} = \{U(f, \epsilon) : f \in C[0,1] \wedge \epsilon > 0\}$ where $U(f, \epsilon) = \{g : g \in C[0,1] \wedge \sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon\}$ is a basis for a topology $\mathcal{T}_2$ on $C[0,1]$
- Prove that $\mathcal{T}_1 \neq \mathcal{T}_2$

Let's generalize the first two questions a bit.

Lemma 1 *Let $\mathcal{B} = \{B(f,\epsilon) : f \in C[0,1] \wedge \epsilon > 0\}$ where $B(f,\epsilon) = \{g : g \in C[0,1] \wedge d(f,g) < \epsilon\}$ such that d is a pseudo-metric, i.e: (1) semi-definiteness $d(a,a) = 0$ (2) symmetry $d(a,b) = d(b,a)$ (3) triangle inequality $d(a,c) \leq d(a,b) + d(b,c)$.*

$\mathcal{B}$ generates a topology $\mathcal{T}$ on $C[0,1]$.

Remark 1 *$f \in B(f,\epsilon)$ for all $f \in C[0,1]$ and $\epsilon > 0$*

Proof of Lemma ??

In order to prove $\mathcal{B}$ generates a topology $\mathcal{T}$ on $C[0,1]$, we need to prove two properties (1) $C[0,1] = \bigcup_{B \in \mathcal{B}} B$ and (2) for any $B_1, B_2 \in \mathcal{B}$, for all $f \in B_1 \cap B_2$, there exists a $B_3 \in \mathcal{B}$ such that $f \in B_3 \subseteq B_1 \cap B_2$

For any $f \in C[0,1]$, $f \in B(f,1) \in \mathcal{B}$. On the other hand, $B(f,\epsilon) \subseteq C[0,1]$ for all $f \in C[0,1]$ and $\epsilon > 0$. Hence, (1)

(2)

Let $B_1 = B(f_1,\epsilon_1)$ and $B_2 = B(f_2,\epsilon_2)$.

For any $f \in B_1 \cap B_2$, we have $d(f_1,f) < \epsilon_1$ and $d(f_2,f) < \epsilon_2$. Hence, choose a positive $\epsilon = \min\{\epsilon_1 - d(f_1,f), \epsilon_2 - d(f_2,f)\}$ and $B_3 = B(f,\epsilon)$

For any $g \in B_3$,

$$\begin{aligned}
d(f_1,g) &\leq d(f_1,f) + d(f,g) && \text{(triangle inequality)} \\
&< d(f_1,f) + \epsilon && (g \in B_3) \\
&\leq d(f_1,f) + (\epsilon_1 - d(f_1,f)) && \text{(choice of } \epsilon\text{)} \\
&= \epsilon_1
\end{aligned}$$

Similarly, $d(f_2,g) < \epsilon_2$. Therefore, $f \in B_3 \subseteq B_1 \cap B_2$, So (2)

Main proof

. Now, we apply Lemma **??** for $\mathcal{M}$ and $\mathcal{U}$.

For any $x \in [0,1]$ by *Triangle inequality*, $|a(x) - c(x)| \leq |a(x) - b(x)| + |b(x) - c(x)|$

From the properties of *Riemann integral*,

$$\int_0^1 |a-c| \leq \int_0^1 |a-b| + \int_0^1 |b-c|$$

Furthermore, the LHS is upper-bounded by the RHS, hence their supremums

$$\sup_{x \in [0,1]} |a(x) - c(x)| \leq \sup_{x \in [0,1]} |a(x) - b(x)| + \sup_{x \in [0,1]} |b(x) - c(x)|$$

In the last question, it is obvious that for any $n > 0$ there exists a function $f \in M(0,1)$ such that $\min_{x \in [0,1]} f(x) = -n$ and $\max_{x \in [0,1]} f(x) = n$. If f is in any member $U(f_2,\epsilon_2)$ of $\mathcal{U}$, it must be that $\epsilon > n$. So, there exists a function $g \in U(f_2,\epsilon_2)$ with $\int_0^1 |0 - g| = n > 1$ or $U(f_2,\epsilon_2) \setminus M(0,1) \neq \varnothing$. By Proposition 2.3.4 in the book, $\mathcal{T}_1 \neq \mathcal{T}_2$

Exercise 3.1.5.v

Exercise 3.2.9

Let S be a dense subset of a topological space $(X, \mathcal{T})$. Prove that for every open subset U of X, $\overline{S \cap U} = \overline{U}$

Lemma 2 *Let A, B be subsets of a topological space $(X, \mathcal{T})$, then $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$*

Proof of Lemma ??

We will prove (1) $A \cap B \subseteq \overline{A} \cap \overline{B}$ and (2) limit points of $A \cap B$ is in $\overline{A} \cap \overline{B}$

(1)

Any point in A is in $\overline{A}$, any point in B is in $\overline{B}$. Then any point in A and B is in $\overline{A}$ and $\overline{B}$. Hence (1)

(2)

Let $x \in X$ be a limit point of $A \cap B$, so any open set containing x contains a point in $A \cap B$. Hence, any open set containing x contains a point in A, that implies x is a limit point of A, $x \in \overline{A}$. Similarly, $x \in \overline{B}$. Therefore, (2)

Main proof

.

Apply Lemma **??**, $\overline{S \cap U} \subseteq \overline{U}$

Now, we will prove that $\overline{U} \subseteq \overline{S \cap U}$, i.e (1) $u \in U \implies u \in \overline{S \cap U}$ and (2) x is limit point of $U \implies x \in \overline{S \cap U}$

(1)

Let $u \in U$. For any open set O containing u,

S is dense, so u is a limit point of S. Given the open set $U \cap O$ containing u, it must also contains a point $s \in S$. Hence $s \in S \cap (U \cap O) = (S \cap U) \cap O$. Therefore, for any open set O containing u, the intersection of $S \cap U$ and O is non-empty by the construction of s. So, u is a limit point of $S \cap U$

(2)

Let $x \in X$ be a limit point of U. For any open set O containing x, take $u \in U \cap O$,

S is dense, so u is a limit point of S. Given the open set $U \cap O$ containing u, it must also contains a point $s \in S$. Hence $s \in S \cap (U \cap O) = (S \cap U) \cap O$. Therefore, for any open set O containing x, the intersection of $S \cap U$ and O is non-empty by the construction of s. So, x is a limit point of $S \cap U$

Exercise 3.2.11.v

Let $\mathcal{B} = \{[a, b) : a \in \mathbb{R}, b \in \mathbb{Q}, a < b\}$, $\mathcal{B}$ is a basis for a topology $\mathcal{T}_1$ on $\mathbb{R}$, namely the *Sorgenfrey line*. Prove that the *Sorgenfrey line* does not satisfy the second axiom of countability, i.e $(\mathbb{R}, \mathcal{T}_1)$ cannot be generated by a countable number of open sets.

TODO

Proposition 3.3.3

The only clopen sets of $\mathbb{R}$ are $\mathbb{R}$ and $\varnothing$

Lemma 3 (Lemma 3.3.2) *Let S be a subset of $\mathbb{R}$ bounded above and let $p = \sup S$. If S is closed, then $p \in S$*

Lemma 4 *Let S be a non-empty subset of $\mathbb{R}$ bounded above and let $p = \sup S$. If S is open, then $p \notin S$*

Proof of Lemma ??

S is open and $p \in S$, we can choose an open interval $(a, b) \subseteq S$ containing p. Hence, $p < b$ and there exists q such that $p < q < b$. Contradiction to the assumption that p is the supremum of S

Main proof

.

Suppose A, B are non-empty clopen sets in $\mathbb{R}$ such that $A \cap B = \varnothing$ and $A \cup B = \mathbb{R}$. Choose $a_1 \in A$ and $b_1 \in B$, without loss of generality, assume that $a_1 < b_1$.

Consider the closed set $S = A \cap [a_1, b_1]$ and its supremum $p = \sup S$

S is closed and bounded above by b_1, by Lemma **??**, $p \in S$. Furthermore, $b_1 \notin A$ implies $b_1 \notin S = A \cap [a_1, b_1]$, we have the strict inequality $p < b_1$

Now we will construct an element $t \in S$ that is greater than p then conclude the contradiction. $p \in S \subseteq A$, A is open then there exists an open interval $(a_2, a_3) \subseteq A$ such that $p \in (a_2, a_3) \subseteq A$. By the strict inequality $p < b_1$, we can choose $t \in (p, \min(a_3, b_1)) \subseteq (a_2, a_3) \subseteq A$. We also have $t \in (p, \min(a_3, b_1)) \subseteq [a_1, b_1]$. Hence, $p < t \in A \cap [a_1, b_1] = S$. Contradiction

Exercise 4.1.11

Let A, B be connected subspaces of a topological space $(X, \mathcal{T})$. If $A \cap B \neq \varnothing$, prove that the subspace $A \cup B$ is connected.

Main proof

.

We will prove the statement by contradiction, first we assume that $A \cap B \neq \varnothing$ and $A \cup B$ is disconnected. Hence, we can find $P, Q \in A \cup B$ such that $P \neq \varnothing$, $Q \neq \varnothing$, $P \cup Q = A \cup B$, and $P \cap Q = \varnothing$. P, Q are corresponding to two open sets in X, namely $P = O_P \cap (A \cup B)$ and $Q = O_Q \cap (A \cup B)$.

Consider 2 pairs of sets: $(O_P \cap A, O_Q \cap A)$ and $(O_P \cap B, O_Q \cap B)$. Claim that there must be at least a pair with no empty set.

(case 1) $(O_P \cap A) = (O_P \cap B) = \varnothing$

$(O_P \cap A) \cup (O_P \cap B) = \varnothing$ implies $P = O_P \cap (A \cup B) = \varnothing$, contradiction

(case 2) $(O_P \cap A) = (O_Q \cap B) = \varnothing$.

$O_P \cap A = \varnothing$ implies that

$$
\begin{aligned}
P &= O_P \cap (A \cup B) \\
&= O_P \cap (A \cup B \setminus A) \\
&= (O_P \cap A) \cup (O_P \cap B \setminus A) \\
&= O_P \cap B \setminus A \subseteq B \setminus A
\end{aligned}
$$

$O_Q \cap B = \varnothing$ implies that

$$
\begin{aligned}
Q &= O_Q \cap (A \cup B) \\
&= O_Q \cap (A \setminus B \cup B) \\
&= (O_Q \cap A \setminus B) \cup (O_Q \cap B) \\
&= O_Q \cap A \setminus B \subseteq A \setminus B
\end{aligned}
$$

Hence, an element $x \in A \cap B$ is not in either P or Q, contradiction

Therefore, there must be at least a pair in $(O_P \cap A, O_Q \cap A)$ and $(O_P \cap B, O_Q \cap B)$ is both non-empty sets. Without loss of generality, assume that $O_P \cap A, O_Q \cap A$ are both non-empty.

$$
\begin{aligned}
(O_P \cap A) \cup (O_A \cap A) &= (O_P \cup O_Q) \cap A \\
&= A && \text{since } A \cup B \subseteq O_P \cup O_Q
\end{aligned}
$$

$$
\begin{aligned}
(O_P \cap A) \cap (O_A \cap A) &= (O_P \cap O_Q) \cap A \\
&= \varnothing && \text{since } O_P \cap O_Q \text{ is outside of } A \cup B
\end{aligned}
$$

Exercise 4.1.15

The closed interval $[a, b]$ for $a, b \in \mathbb{R}$ is connected.

Let's recognize all the open sets and closed sets in $[a, b]$ first. O is an open set in $[a, b]$ if and only if $O = O_{\mathbb{R}} \cap [a, b]$ for an open set $O_{\mathbb{R}}$ in $\mathbb{R}$. C is a closed set in $[a, b]$ if and only if $C = [a, b] \setminus O_{\mathbb{R}}$ for an open set $O_{\mathbb{R}}$ in $\mathbb{R}$

Lemma 5 *If C is a closed set in $[a, b]$, C is also a closed set in $\mathbb{R}$.*

Proof of Lemma ??

$\mathbb{R} \setminus C = (\mathbb{R} \setminus [a, b]) \cup O_{\mathbb{R}}$, union of two open sets in $\mathbb{R}$

Main proof

.

Similar to exercise 4.1.11, let prove the statement by contradiction. Let A, B be clopen sets in $[a, b]$ such that $A \cap B = \varnothing$ and $A \cup B = [a, b]$. Choose $a_1 \in A$ and $a_2 \in B$, without loss of generality, assume the strict inequality $a < a_1 < b_1 < b$ since A, B cannot be singleton sets.

A is a closed set in $\mathbb{R}$, consider the closed set $S = A \cap [a_1, b_1]$ in $[a, b]$ and its supremum $p = \sup S$ in $\mathbb{R}$.

S is closed and bounded above by b_1 in $\mathbb{R}$, by Lemma **??**, $p \in S$. Furthermore, $b_1 \notin A$ implies $b_1 \notin S = A \cap [a_1, b_1]$, we have the strict inequality $a < a_1 \leq p < b_1 \leq b$

Now we will construct an element $t \in S$ that is greater than p then conclude the contradiction. $p \in S \subseteq A = O_A \cap [a, b] = (O_A \cap (a, b)) \cup (O_A \cap \{a, b\})$ where O_A is an open set in $\mathbb{R}$. $a < p < b$ implies p in the open set $O_A \cap (a, b) \subseteq A$ in $\mathbb{R}$. By the strict inequality $p < b_1$, we can choose $t \in (p, \min(a_3, b_1)) \subseteq (a_2, a_3) \subseteq O_A \cap (a, b) \subseteq A$. We also have $t \in (p, \min(a_3, b_1)) \subseteq [a_1, b_1]$. Hence, $p < t \in A \cap [a_1, b_1] = S$. Contradiction

Comment: $O_A \cap (a, b)$ is the interior of $A = O_A \cap [a, b]$ in $\mathbb{R}$

Exercise 4.1.17.v

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Define a set $C \subseteq \mathbb{R}$ to be closed if $C = A \cup T$ where A is closed in $\mathbb{R}$ and $T \subseteq S$. The complements of these closed sets form a topology $\mathcal{T}$ on $\mathbb{R}$ which is Hausdorff but not regular.

Definition 12 (Hausdorff space or T_2-space) *A topological space $(X, \mathcal{T})$ is said to be Hausdorff (or T_2-space) if given any pair of distinct points a, b in X there exist open sets A, B such that $a \in A$, $b \in B$, and $A \cap B = \varnothing$*

Definition 13 (Regular space) *A topological space $(X, \mathcal{T})$ is said to be regular space if any closed set A and any point $x \in X \setminus A$, there exist open sets U, V such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.*

Lemma 6 *For any index set J and $A_j \cap B_j = \varnothing$ for all $j \in J$*

$$\bigcap_{j \in J} A_j \cup B_j = \bigcup_{J_A \in \mathcal{P}(J)} \left[\left(\bigcap_{j \in J_A} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A} B_j \right) \right]$$

Proof of Lemma ??

Proof

For all $x \in X = \bigcap_{j \in J} A_j \cup B_j$, for each $j \in J$, x must be either in A_j or B_j. Let $J_A(x) = \{j : j \in J, x \in A_j\} \subseteq \mathcal{P}(J)$ be the set of indices where $x \in A_j$ and let $J_B(x) = J \setminus J_A(x)$. So that

$$x \in \left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right)$$

On the other hand, $\bigcap_{j \in J_A} A_j \subseteq \bigcap_{j \in J_A} A_j \cup B_j$ and $\bigcap_{j \in J \setminus J_A} B_j \subseteq \bigcap_{j \in J \setminus J_A} A_J \cup B_j$, we have

$$\begin{aligned}
\left(\bigcap_{j \in J_A(x)} A_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(x)} B_j \right) &\subseteq \left(\bigcap_{j \in J_A(y)} A_j \cup B_j \right) \cap \left(\bigcap_{j \in J \setminus J_A(y)} A_j \cup B_j \right) \\
&= \bigcap_{j \in J} A_j \cup B_j \\
&= X
\end{aligned}$$

$\mathcal{T}$ is a topology

Take $A = \varnothing$ and $T = \varnothing$, then $C = A \cup T = \varnothing$. So $\mathbb{R}$ is an open set in $\mathcal{T}$. Take $A = \mathbb{R}$, then $C = A \cup T = \mathbb{R}$. So $\varnothing$ is an open set in $\mathcal{T}$

$$\begin{aligned}
\mathbb{R} \setminus C_1 \cap \mathbb{R} \setminus C_2 &= \mathbb{R} \setminus (A_1 \cup T_1) \cap \mathbb{R} \setminus (A_2 \cup T_2) \\
&= \mathbb{R} \setminus ((A_1 \cup T_1) \cup (A_2 \cup T_2)) \\
&= \mathbb{R} \setminus ((A_1 \cup A_2) \cup (T_1 \cup T_2))
\end{aligned}$$

$A_1 \cup A_2$ is an closed set in the euclidean topology, $T_1 \cup T_2 \subseteq S$. Hence the intersection of two open sets in $\mathcal{T}$ is an open set in $\mathcal{T}$

Let J be an index set, a union of open sets in $\mathcal{T}$ has the form

$$\bigcup_{j \in J} \mathbb{R} \setminus (A_j \cup T_j) = \mathbb{R} \setminus \bigcap_{j \in J} A_j \cup T_j$$

We need to prove that $C = \bigcap_{j \in J} A_j \cup T_j$ can be written in the form $A \cup T$ where A is a closed set in the euclidean topology and $T \subseteq S$. Invoke the lemma **??**

$$
\begin{aligned}
\bigcap_{j\in J} A_j \cup T_j &= \bigcap_{j\in J} A_j \cup T_j \setminus A_j \\
&= \bigcup_{J_A \in \mathcal{P}(J)} \left[\left(\bigcap_{j\in J_A} A_j \right) \cap \left(\bigcap_{j\in J\setminus J_A} T_j \setminus A_j \right) \right]
\end{aligned}
$$

We split $\mathcal{P}(J)$ into two groups: (1) $\{J\} \in \mathcal{P}(J)$ and (2) $\mathcal{P}(J) \setminus \{J\}$.

$$
\bigcap_{j\in J} A_j \cup T_j = \bigcap_{j\in J} A_j \cup \bigcup_{J_A \in \mathcal{P}(J)\setminus J} \left[\left(\bigcap_{j\in J_A} A_j \right) \cap \left(\bigcap_{j\in J\setminus J_A} T_j \setminus A_j \right) \right]
$$

The set $\bigcap_{j\in J} A_j$ is a union of closed set in euclidean space hence a closed set in euclidean space.

We further have

$$
\left(\bigcap_{j\in J_A} A_j \right) \cap \left(\bigcap_{j\in J\setminus J_A} T_j \setminus A_j \right) \subseteq \bigcap_{j\in J\setminus J_A} T_j \setminus A_j \subseteq S
$$

for all $J_A \in \mathcal{P} \setminus J$ since $J \setminus J_A$ is non-empty. Take A and T as follows

$$
\begin{aligned}
A &= \bigcap_{j\in J} A_j \\
T &= \bigcup_{J_A \in \mathcal{P}(J)\setminus J} \left[\left(\bigcap_{j\in J_A} A_j \right) \cap \left(\bigcap_{j\in J\setminus J_A} T_j \setminus A_j \right) \right]
\end{aligned}
$$

So, $C = A \cup T$ which is a union of a closed set in the euclidean topology and a subset of S

$\mathcal{T}$ is Hausdorff

Any closed set in the euclidean topology is closed in $\mathcal{T}$ by taking $T = \varnothing$ in the form of closed set in $\mathcal{T}$: $C = A \cup T$. Hence, any open set in the euclidean topology is open in $\mathcal{T}$.

Given any two point $a < b \in \mathbb{R}$, take $A = (-\infty, \frac{a+b}{2})$ and $B = (\frac{a+b}{2}, +\infty)$

$\mathcal{T}$ is not regular

We will construct an example where it is not able to construct U, V

Suppose that $\mathcal{T}$ is a regular space, let $A = S = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $x = 0$. There must be an open set V that contains 0 but not S. All open sets in $\mathcal{T}$ has the form

$$O = \mathbb{R} \setminus (A \cup T) = \mathbb{R} \setminus A \cap \mathbb{R} \setminus T$$

where A is a closed set in the euclidean topology and $T \subseteq S$.

$0 \in \mathbb{R} \setminus A$ an open set in the euclidean topology, there must be an open interval (a, b) such that $a < 0 < b$, hence that interval will contain some element of S, i.e: $\{\frac{1}{n} : n \in \mathbb{N}, n > \frac{1}{b}\}$

$\mathcal{T}$ is a topology (shorten approach)

I came across a short reasoning for the last statement [1]

$C = \bigcap_{i \in I} A_i \cup T_i$ where each A_i is Euclidean-closed and each $T_i \subseteq S$. Let $A = \bigcap_{i \in I} A_i$; certainly A is Euclidean-closed.

If $x \in C \setminus A$, there exists A_i such that $x \notin A_i$. But $x \in C$, so $x \in A_i \cup T_i$. Therefore, $x \in T_i \subseteq S$

Exercise 4.2.8

Let $(X, \mathcal{T})$ be a discrete topological space. Prove that $(X, \mathcal{T})$ is homeomorphic to a subspace of $\mathbb{R}$ if and only if X is countable.

TODO

Exercise 4.3

Definition 14 *Let X be a unit circle in $\mathbb{R}^2$*

$$X = \{\langle x, y\rangle : x^2 + y^2 = 1\}$$

Let Y be two disjoint circles in $\mathbb{R}^2$

$$Y = \{\langle x, y\rangle : x^2 + y^2 = 1\} \cup \{\langle x, y\rangle : (x-2)^2 + y^2 = 1\}$$

Let Z be two intersecting circles in $\mathbb{R}^2$

$$Z = \{\langle x, y\rangle : x^2 + y^2 = 1\} \cup \{\langle x, y\rangle : (x-3/2)^2 + y^2 = 1\}$$

Definition 15 (Sorgenfrey Line) *Let $\mathcal{B} = \{[a, b) : a \in \mathbb{R}, b \in \mathbb{Q}, a < b\}$. The set generated by $\mathcal{B}$ is a topology on $\mathbb{R}$*

[1] https://math.stackexchange.com/a/65558/700122

Lemma 7 *If f is a homeomorphism from $(A, \mathcal{T}^A)$ to $(B, \mathcal{T}^B)$ and $A_1 \subseteq A$. Let $B_1 = f(A_1)$ be the image of A_1 over f. Define $f_1 : A_1 \to B_1$ with $f_1(a) = f(a)$. f_1 is a homeomorphism from $(A_1, \mathcal{T}^A_{A_1})$ to $(B_1, \mathcal{T}^B_{B_1})$*

Proof of Lemma ??

f_1 is bijective.

Any open set in A_1 has the form $O_{A_1} = A_1 \cup O_A$ where O_A is an open set in A. So $f_1(O_{A_1}) = f(O_{A_1}) = f(A_1) \cup f(O_A)$ since f is injective. Furthermore, f is an homeomorphism, $f_1(O_{A_1}) = B_1 \cup O_B$. So, $f(O_A)$ is an open set in B_1. Similar argument for f_1^{-1}

Main proof

.

main proof

(3.i) $X \setminus \{\langle 1, 0\rangle\}$ is homeomorphic to the open interval $(0, 1)$ by the homeomorphism $f(\langle x, y\rangle) =$ arc length [2] from $\langle 1, 0\rangle \to \langle x, y\rangle$ divided by 2π

(3.ii) $X \ncong (0, 1)$

Suppose $X \cong (0, 1)$ by homeomorphic f that maps $\langle 1, 0\rangle \mapsto a$, by remark 4.3.6, $(0, 1) \cong X \setminus \{\langle 1, 0\rangle\} \cong (0, a) \cup (a, 1)$ where $(0, a)$ and $(a, 1)$ be non-empty. The left most is a connected and the right most not a disconnected.

(3.ii) $[0, 1] \ncong X$

Suppose $[0, 1] \cong X$ by homeomorphic f that maps $0 \mapsto a$ and $1 \mapsto b$ where $a \neq b$. By remark 4.3.6, $[0, 1] \cong X \implies (0, 1] \cong X \setminus \{a\} \implies (0, 1) \cong X \setminus \{a, b\}$. The RHS consists of two segments $a \to b$ and $b \to a$ each of which is homeomorphic to open intervals in $\mathbb{R}$. So, $(0, 1) \cong X \setminus \{a, b\} = (a \to b) \cup (b \to a) \cong (0, 1) \cup (1, 2)$. The left most is a connected and the right most not a disconnected.

(3.iii) $[0, 1) \ncong X$

Suppose $[0, 1) \ncong X$ by homeomorphic f that maps $1/2 \mapsto a$. By remark 4.3.6, $[0, 1) \ncong X \implies [0, 1/2) \cup (1/2, 1) \cong X \setminus \{a\} \cong (0, 1)$. The left most is a disconnected and the right most not a connected.

(3.iv) X is not homeomorphic to any interval

Same argument in (3.iii)

(4.i) $Y \ncong X$

Y is disconnected and X is connected

(4.ii) Y is not homeomorphic to any interval

Y is disconnected and any interval is connected

(5.i) Z is not homeomorphic to any interval

Let a, b be the two intersecting points in Z, $Z \setminus \{a, b\}$ consists of 4 segments each of which is homeomorphic to open intervals in $\mathbb{R}$. Where $f(a), f(b)$ split an interval into 3 intervals where f is a homeomorphism from Z to an interval.

(5.ii) Z is not homeomorphic to X

[2] counter-clockwise

Same argument. Where $f(a), f(b)$ split X into 2 intervals where f is a homeomorphism from Z to X.

(5.ii) Z is not homeomorphic to Y

Same argument. Where $f(a), f(b)$ split X into 2 or 3 intervals where f is a homeomorphism from Z to X.

(6) Sorgenfrey line $\mathcal{T}$ is not homeomorphic to $\mathbb{R}, \mathbb{R}^2$ or any subspace of either of these spaces

Sorgenfrey line is disconnected by

$$\mathbb{R} = \left(\bigcup_{n=1}^{\infty} [-n, 0) \right) \cup \left(\bigcup_{n=1}^{\infty} [0, n) \right)$$

While $\mathbb{R}$ is connected, so $\mathcal{T} \not\cong \mathbb{R}$

Suppose $\mathbb{R}^2 \cong \mathcal{T}$ by $f : \mathbb{R}^2 \to \mathbb{R}$. Let $A = \{\langle x, 0 \rangle : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be the horizontal line at $y = 0$. By lemma **??**, $A \cong \mathcal{T}_{f(A)}$. Furthermore, $A \cong \mathbb{R}$, so $\mathbb{R} \cong \mathcal{T}_{f(A)}$.

We will now prove that $f(A)$ is disconnected

Let $a < b < c \in f(A)$ be 3 distinct points since $f(A)$ has at least 3 points. $f(A)$ is disconnected by

$$f(A) = \left[f(A) \cap \left(\bigcup_{n=1}^{\infty} [-n, b) \right) \right] \cup \left[f(A) \cap \left(\bigcup_{n=1}^{\infty} [b, n) \right) \right]$$

where the left set has at least one element, namely a hence non-empty. Similarly for the right set

Exercise 4.3.7.iii

$\mathcal{T}_2$ consists of $\mathbb{R}, \varnothing$ and every interval $(-r, +r)$ for all positive real number r. $\mathcal{T}_9$ consists of $\mathbb{R}, \varnothing$ and every interval $(-r, +r)$ and $[-r, +r]$ for all positive real number r

Is $\mathcal{T}_2 \cong \mathcal{T}_9$?

TODO

Exercise 4.3.8

Let $(X, \mathcal{T})$ be a topological space where X is an infinite set.

(i)* $(X, \mathcal{T})$ has a subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_1)$ where either $\mathcal{T}_1$ is the indiscrete topology or $(\mathbb{N}, \mathcal{T}_1)$ is a T_0-space

(ii)** Let $(X, \mathcal{T})$ be a T_1-space. Then $(X, \mathcal{T})$ has a subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_2)$ where $\mathcal{T}_2$ is either the finite-closed topology or the discrete topology

(iii) Deduce from (ii) that any infinite Hausdorff space contains an infinite discrete subspace and hence a subspace homeomorphic to $\mathbb{N}$ with the discrete topology (iv)** Let $(X, \mathcal{T})$ be a T_0-space which has no infinite T_1-subspaces. Then the space $(X, \mathcal{T})$ has a subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_3)$ where $\mathcal{T}_3$ an initial segment topology or a final segment topology.

(v) Deduce from the above that every infinite topological space has a subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_4)$ where $\mathcal{T}_4$ is the indiscrete topology, the discrete topology, the finite-closed topology, initial segment topology or the final segment topology. Further, no two of these five topology on $\mathbb{N}$ are homeomorphic

Lemma 8 *Distinguishability is preserved under subspace, i.e if A is a subspace of X, a and b are in distinguishable in X implies a and b are indistinguishable in A.*

Lemma 9 *Indistinguishability is an equivalent relation.*

Lemma 10 *Suppose $A \subseteq X$ and $|A| \geq 2$, A consists of pairwise indistinguishable points if and only if A is indiscrete*

Lemma 11 *T_1-space is preserved under subspace, i.e if X is a T_1-space then $A \subseteq X$ a subspace of X is also T_1*

Lemma 12 *If X is a T_1-space but not a finite-closed topological space, X has an infinite closed proper subset A.*

Lemma 13 *T_2-space (Hausdorff) is preserved under subspace, i.e if X is a T_2-space then $A \subseteq X$ a subspace of X is also T_2*

Proof of Lemma ??

Proof of Lemma ??

Main proof

.

(i)

Suppose $(X, \mathcal{T})$ has no subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_1)$ that is an indiscrete space. We will prove that $(X, \mathcal{T})$ must have a subspace homeomorphic to $(\mathbb{N}, \mathcal{T}_1)$ that is T_0.

The premise implies that all indistinguishable subsets of X is finite, namely $S_1, S_2, ...$ If number of indistinguishable subsets of X is infinite. Invoke AC, choose each $x_i \in S_i$ for all $i \in \mathbb{N}$. The set $\{x_i : i \in \mathbb{N}\}$ is T_0 since S_i are disjoint indistinguishable subsets. Otherwise, number of indistinguishable subsets of X is finite and these sets are finite imply that $X \setminus \bigcup_{i \in \mathbb{N}} S_i$ is infinite. A countably infinite subset of this set is T_0

(ii)

Let an infinite set X be a T_1-space, suppose there is no countably infinite subset of X that is finite-closed. We will construct an infinite subset of X that is discrete.

The premise implies if A is a countably infinite subset of X then A is not finite-closed.

Induction step

Let A_1 be a countably infinite closed proper subset of X. By the premise and lemma **??**, A_1 is a T_1-space and not finite-closed.

By lemma **??**, A_1 has a countably infinite closed proper subset, namely A_2. By the premise and lemma **??**, A_2 is also a T_1-space and not finite-closed.

By induction, we can construct an infinite sequence of countably subsets

$$X \supset A_1 \supset A_2 \supset A_3 \supset ...$$

where A_{i+1} is a countably infinite closed proper subset of A_i. We have $A_i \setminus A_{i+1} \neq \varnothing$ is open in A_i, so

$$\begin{aligned}
A_i \setminus A_{i+1} &= O_{i-1} \cap A_i && \text{where } O_{i-1} \text{ is open in } A_{i-1} \\
&= (O_{i-2} \cap A_{i-1}) \cap A_i && \text{where } O_{i-2} \text{ is open in } A_{i-2} \\
&= ... \\
&= O \cap A_1 \cap A_2 \cap ... \cap A_{i-1} \cap A_i && \text{where } O \text{ is open in } X \\
&= O \cap A_i \\
&= O^{(i)} \cap A_i && O^{(i)} = O
\end{aligned}$$

We observe that, for each $i \in \mathbb{N}$, there exists an open set $O^{(i)}$ of X that contains $A_i \setminus A_{i+1}$ but does not contain A_{i+1}

Invoke AC, we choose x_i from each disjoint set $A_i \setminus A_{i+1}$ for all $i \in \mathbb{N}$. For each x_i, there exists an open set $O^{(i)}$ containing x_i and does not contain $\{x_{i+1}, x_{i+2}, ...\} \subseteq A_{i+1}$.

X is T_1 implies that every finite subset of X is closed. Therefore, $F_i = X \setminus \{x_1, x_2, ..., x_{i-1}\}$ is open. So, the open set $O^{(i)} \cap F_i$ does not contain $\{x_1, x_2, ..., x_{i-1}\}$ and $\{x_{i+1}, x_{i+2}, ...\}$. Therefore, $\{x_i\}$ is an open set on $\{x_1, x_2, ...\}$. Every singleton set is open, so $\{x_1, x_2, ...\}$ is discrete.

(iii)

Let X be an Hausdorff space and A be any infinite subspace of X. Let $a \in A$, For any other point $b \in A$, since X is Hausdorff, there exists an open set $U_b \subseteq X$ such that $a \notin U_b$ and $b \in U_b$. The open set $U^{(a)} \subseteq X$ is defined as follows

$$U^{(a)} = \bigcup_{b \in X \setminus \{a\}} U_b$$

Furthermore, $U^{(a)} \cap X = \{a\}$ implies that every singleton set in A is closed. Hence, every subset of X is a T_1-space.

From (ii), there exists an countably infinite subset of X that is either finite-closed or discrete. Now, we will prove that every infinite subset of X is not finite-closed.

Suppose A is an infinite subset of X that is finite-closed. Let $x, y \in A$, since X is a Hausdorff space, there exist two disjoint open set $U_x, U_y \subseteq X$ such that $x \in U_x$ and $y \in U_y$. $V_x = A \cap U_x$ and $V_y = A \cap U_y$ are two non-empty open sets in the subspace A. Two disjoint open sets V_y, V_y in A must have $V_y \subseteq A \setminus V_y$. $A \setminus V_x$ is closed in A, so $A \setminus V_x$ is finite, so V_y is finite. That implies $A \setminus V_y$ is both closed and infinite, contradicts with the premise.

(iv)

TODO

Exercise 4.3.9.iii

Prove that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ is a local homeomorphism, then f maps every open set of X into an open set of Y

Exercise 5.2.6

An analysis problem using topology

Lemma 14 *If A is a connected subspace of $(X, \mathcal{T})$ and $A \subseteq B \subseteq \overline{A}$, then B is connected.*

(i) Show that the subspace

$$Y = \{\langle x, y \rangle : y = \sin(1/x), 0 < x \leq 1\}$$

of $\mathbb{R}^2$ is connected

(ii) Verify that

$$\overline{Y} = Y \cup \{\langle 0, y \rangle : -1 \leq y \leq 1\}$$

(iii) From lemma **??**, $\overline{Y}$ is connected.

Lemma 15 *For two distinct points $x_1, x_2 \in \mathbb{R}$*

$$|\sin(x_2) - \sin(x_1)| < |x_2 - x_1|$$

Proof of Lemma ??

Suppose that B is not connected, i.e there exists two non-empty open sets P, Q in $\mathcal{T}$ that split B into two parts. Since A is connected, A must be in either P or Q. Let $A \in P$, Q is non-empty so there exists a limit point of A in Q, namely q. q is a limit point of A, so there is no open sets in $\mathcal{T}$ containing x and not containing A. Contradiction

Main proof

.

(i)

We will use the Lemma 5.1.2 state that a mapping f from $(X, \mathcal{T})$ into $(Y, \mathcal{T}')$ is continuous if for each $a \in X$ and each $U \in \mathcal{T}'$ with $f(a) \in U$, there exists a $V \in \mathcal{T}$ such that $a \in V$ and $f(V) \subseteq U$

Consider the mapping $f : (0, 1] \to \mathbb{R}^2$ that maps $x \mapsto \langle x, \sin(1/x)\rangle$

For each $a \in (0, 1]$, $f(a) = \langle a, \sin(1/a)\rangle$ and each open set containing $f(a)$. Every open set U containing $f(a)$ in $\mathbb{R}^2$ is a union of open squares, so there exists an open square containing $f(a)$. Now we choose an open square O_U centered at $f(a)$ and contained in the previous open square. Let 2ϵ be the size of O_U such that $0 < \epsilon < a$, so

$$O_U = (a - \epsilon, a + \epsilon) \times (\sin\frac{1}{a} - \epsilon, \sin\frac{1}{a} + \epsilon)$$

Now we will construct an open set $O_V \subseteq (0, 1]$.

Let $a > \delta > 0$, for all $x \in (a - \delta, a + \delta)$

$$\begin{aligned}
\left|\sin\frac{1}{x} - \sin\frac{1}{a}\right| &< \left|\frac{1}{x} - \frac{1}{a}\right| && \text{(lemma ??)}\\
&= \frac{|a - x|}{xa} && (a > 0 \text{ and } x > 0)\\
&< \frac{\delta}{(a - \delta)a} && (a > 0 \text{ and } x > 0)\\
&= \frac{1}{(\frac{a}{\delta} - 1)a}
\end{aligned}$$

We want to choose δ such that $\sin\frac{1}{x}$ stays within $(\sin\frac{1}{a} - \epsilon, \sin\frac{1}{a} + \epsilon)$, i.e

$$\left|\sin\frac{1}{x} - \sin\frac{1}{a}\right| < \epsilon$$

Let

$$\begin{aligned}
\frac{1}{(\frac{a}{\delta} - 1)a} &< \epsilon\\
(\frac{a}{\delta} - 1)a &> \frac{1}{\epsilon}\\
\frac{a}{\delta} - 1 &> \frac{1}{a\epsilon}\\
\frac{a}{\delta} &> \frac{1}{a\epsilon} + 1\\
\delta &< \frac{a}{\frac{1}{a\epsilon} + 1}
\end{aligned}$$

Therefore, for any $a \in (0, 1]$, for any open set $U \in \mathbb{R}^2$ containing $f(a)$, there exists an open square O_U of size 2ϵ centered at $f(a)$ such that $O_U \subseteq U$ and $a > \epsilon > 0$. Let the open interval $V = (a - \delta, a + \delta)$ where $\delta < \frac{a}{\frac{1}{a\epsilon}+1}$ and $\delta < \epsilon$. Then $a \in V$ and

$$f(V) \subseteq (a - \delta, a + \delta) \times (a - \epsilon, a + \epsilon) \subseteq O_U \subseteq U$$

Hence, f is continuous. Furthermore, $(0, 1]$ is connected, so $Y = f((0, 1])$ is also connected.

(ii)

We will prove precisely that $Z = \{\langle 0, y\rangle : -1 \leq y \leq 1\}$ is the set of limit points not contained in Y.

We split $\mathbb{R}^2 \setminus Y$ into several disjoint regions

$$\begin{aligned} A_1 &= \{\langle x, y\rangle : x > 1\} \\ A_2 &= \{\langle x, y\rangle : 0 < x\} \\ A_3 &= \{\langle x, y\rangle : 0 < x \leq 1\} \setminus Y \\ A_4 &= \{\langle 0, y\rangle : 1 < y\} \\ A_5 &= \{\langle 0, y\rangle : y < -1\} \\ Z &= \{\langle 0, y\rangle : -1 \leq y \leq 1\} \end{aligned}$$

A_1

For every point $\langle x, y\rangle \in A_1$, the open set $(1, x + 1) \times (y - 1, y + 1)$ contains $\langle x, y\rangle$ but not any point in Y

A_2

For every point $\langle x, y\rangle \in A_2$, the open set $(x - 1, 0) \times (y - 1, y + 1)$ contains $\langle x, y\rangle$ but not any point in Y

A_4

For every point $\langle x, y\rangle \in A_4$, the open set $(-1, 1) \times (1, y + 1)$ contains $\langle x, y\rangle$ but not any point in Y

A_5

For every point $\langle x, y\rangle \in A_5$, the open set $(-1, 1) \times (y - 1, -1)$ contains $\langle x, y\rangle$ but not any point in Y

A_3

For every point $\langle x, y\rangle \in A_3$, let $x > \delta > 0$, consider $x_1 \in (x - \delta, x + \delta) \subseteq (1, 0]$. Distance between $\langle x, y\rangle$ and $\langle x_1, \sin \frac{1}{x_1}\rangle \in Y$ is r where

$$r^2 = \left(y - \sin \frac{1}{x_1}\right)^2 + (x - x_1)^2$$

We will prove that r^2 is bounded below by some positive number.

$$
\begin{aligned}
\left|y-\sin\frac{1}{x_1}\right| &\geq \left|y-\sin\frac{1}{x}\right| - \left|\sin\frac{1}{x_1}-\sin\frac{1}{x}\right| && \text{(Triangle inequality)}\\
&> \left|y-\sin\frac{1}{x}\right| - \left|\frac{1}{x_1}-\frac{1}{x}\right| && \text{(lemma ??)}\\
&= \left|y-\sin\frac{1}{x}\right| - \frac{|x-x_1|}{x_1 x}\\
&> \left|y-\sin\frac{1}{x}\right| - \frac{\delta}{(x-\delta)x}\\
&= \left|y-\sin\frac{1}{x}\right| - \frac{1}{(\frac{x}{\delta}-1)x}
\end{aligned}
$$

Choose $\delta > 0$ small enough such that $\left|y-\sin\frac{1}{x}\right| - \frac{1}{(\frac{x}{\delta}-1)x} > 0$, i.e

$$
\begin{aligned}
\left|y-\sin\frac{1}{x}\right| - \frac{1}{(\frac{x}{\delta}-1)x} &> 0\\
\frac{1}{(\frac{x}{\delta}-1)x} &< \left|y-\sin\frac{1}{x}\right|\\
\frac{1}{\frac{x}{\delta}-1} &< x\left|y-\sin\frac{1}{x}\right|\\
\frac{x}{\delta}-1 &> \frac{1}{x\left|y-\sin\frac{1}{x}\right|}\\
\frac{x}{\delta} &> 1+\frac{1}{x\left|y-\sin\frac{1}{x}\right|}\\
\delta &< \frac{x}{1+\frac{1}{x\left|y-\sin\frac{1}{x}\right|}}
\end{aligned}
$$

we will have

$$
r^2 > \left(\left|y-\sin\frac{1}{x}\right| - \frac{1}{(\frac{x}{\delta}-1)x}\right)^2 > 0
$$

The open disc of radius $\left|y-\sin\frac{1}{x}\right| - \frac{1}{(\frac{x}{\delta}-1)x}$ centered at $\langle x, y\rangle$ does not contain any point in Y

Z

For every point $\langle 0, y\rangle \in Z$, any open set containing $\langle 0, y\rangle$ contains an open disc of radius r centered at $\langle 0, y\rangle$. We will prove that for all $y \in [-1, 1]$ and $0 < r < 1$, there exists a point $\langle x, y\rangle$ in Y that is contained in the open disc, i.e $x < r$

The set of real values $t > 0$ such that $\sin(t) = y$ is $T = \{\arcsin(y) + 2\pi k : k \in \mathbb{Z}\} \cap [0, \infty)$. For any number M, there exists $t \in T$ such that $t > M$.

Let $M = \frac{1}{r}$, there exists $t \in T$ such that $t > \frac{1}{r}$. Let $x = \frac{1}{t} < r$, $\langle x, y\rangle = \langle x, \sin\frac{1}{x}\rangle$

Therefore, every point in Z is a limit point of Y.

Hence, $\overline{Y} = Y \cup Z$ is connected.

Exercise 5.2.7

Let E be the set of all points in $\mathbb{R}^2$ having both coordinates rational. Prove that the space $\mathbb{R}^2 \setminus E$ is path-connected

Main proof

.

Let $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ be two points on $\mathbb{R}^2 \setminus E$. Without loss of generality, assume either one of these two cases occurs: (1) x_1, x_2 irrational or (2) x_1, y_2 irrational.

Case (1) Let $y \in \mathbb{R}$ irrational, f is defined as [3]

$$f_1 : \left[0 \to \frac{1}{3}\right] \mapsto (\langle x_1, y_1\rangle \to \langle x_1, y\rangle) \subseteq \mathbb{R}^2 \setminus E$$
$$f_2 : \left[\frac{1}{3} \to \frac{2}{3}\right] \mapsto (\langle x_1, y\rangle \to \langle x_2, y\rangle) \subseteq \mathbb{R}^2 \setminus E$$
$$f_3 : \left[\frac{2}{3} \to 1\right] \mapsto (\langle x_2, y\rangle \to \langle x_2, y_2\rangle) \subseteq \mathbb{R}^2 \setminus E$$

Case (2) f is defined as

$$f_1 : \left[0 \to \frac{1}{2}\right] \mapsto (\langle x_1, y_1\rangle \to \langle x_1, y_2\rangle) \subseteq \mathbb{R}^2 \setminus E$$
$$f_2 : \left[\frac{1}{2} \to 1\right] \mapsto (\langle x_1, y_2\rangle \to \langle x_2, y_2\rangle) \subseteq \mathbb{R}^2 \setminus E$$

Exercise 5.2.8

Let C be any countable subset of $\mathbb{R}^2$. Prove that space $\mathbb{R}^2 \setminus C$ is path-connected.

TODO

[3] $a \to b$ denotes the directed line segment from a to b

Exercise 5.2.12

Let A and B be subsets of a topological space $(X, \mathcal{T})$. If A and B are both closed, and $A \cup B$ and $A \cap B$ are both connected, show that A and B are connected.

Main proof

.

Suppose A is disconnected, i.e there exists non-empty disjoint closed sets P, Q of A such that $P \cup Q = A$.

Since P is closed in A, $P = C_P \cap A$ where C_P is a closed set in X. Moreover, A is closed, so P is closed in X. Similarly, C is closed in X.

Since $A \cap B$ is connected, $A \cap B$ must be contained in either P or Q.

Without loss of generality, $A \cap B \subseteq Q$

We have two disjoint closed sets P and $Q \cup B$. Contradiction

Exercise 5.2.13.vii

A subset of $\mathbb{R}$ is zero-dimensional if and only if it is totally disconnected.

Main proof

.

($\Rightarrow$)

Suppose a subset $X \subseteq \mathbb{R}$ is zero-dimensional but not totally disconnected, i.e there exists a connected subset $Y \subseteq X$ containing at least two elements.

Let $a, b \in Y \subseteq X$ such that $a < b$, let $A = (-\infty, \frac{a+b}{2})$ and $B = (\frac{a+b}{2}, \infty)$ open in $\mathbb{R}$, so $A \cap X$ is open in X. Since X is zero-dimensional, there exists an clopen set $U \subseteq A$ of A containing a and not containing b. $U \cap Y$ and $(A \setminus U) \cap Y$ are two open sets in Y that separate a and b. So Y is disconnected, contradiction.

Therefore, all connected subsets of X is singleton sets.

($\Leftarrow$)

TODO

Exercise 5.3.15

Same as 5.2.8

Tychonoff Theorem

[4]

Definition 16 (Product Topology) *Let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces where I is an index set. The Cartesian product of the family of sets $\{X_i : i \in I\}$ is denoted by $\prod_{i \in I} X_i$ consists of the set of all functions $f : I \to \bigcup_{i \in I} X_i$ such that $f(i) = x_i \in X_i$. The product space is denoted by $\prod_{i \in I}(X_i, \mathcal{T}_i)$ on the product set $\prod_{i \in I} X_i$ with the product topology $\mathcal{T}$ having the basis*

$$\mathcal{B} = \left\{ \prod_{i \in I} O_i : O_i \in \mathcal{T}_i \text{ and } O_i = X_i \text{ for all but a finite number of } i \right\}$$

Theorem 1 (Tychonoff Theorem) *Let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces where I is an index set. Then $(X, \mathcal{T}) = \prod_{i \in I}(X_i, \mathcal{T}_i)$ is compact if and only if each $(X_i, \mathcal{T}_i)$ is compact.*

Definition 17 (Finite Intersection Property) *Let X be a set and $\mathcal{F}$ be a family of subsets of X. Then $\mathcal{F}$ is said to have the finite intersection property (FIP) if for any finite number $F_1, F_2, ..., F_n$ of elements of $\mathcal{F}$, $\bigcap_{i=1}^{n} F_i \neq \varnothing$*

Axiom 1 (Zorn lemma) *Let $(X, \leq)$ be a non-empty partially ordered set in which every subset which is linearly ordered has an upper bound. Then $(X, \leq)$ has a maximal element.*

Proposition 2 (Compactness) *A topological space $(X, \mathcal{T})$ is compact if and only if every family $\mathcal{F}$ of closed subsets of X with FIP satisfies $\bigcap_{F \in \mathcal{F}} \neq \varnothing$*

Lemma 16 *Let X be a set and $\mathcal{F}$ be a family of subsets of X with FIP. Then there is a maximal family of subsets of X containing $\mathcal{F}$ with FIP.*

Proof of Lemma ??

Let Z be the collection of all families of subsets of X that have FIP. Consider the order on Z, that is, $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let Y be any linearly ordered collection of elements of Z. $\bigcup_{\mathcal{Y} \in Y} \mathcal{Y}$ contains all $\mathcal{Y} \in Y$ with FIP, so $\bigcup_{\mathcal{Y} \in Y} \mathcal{Y}$ is an upper bound of Y. By Zorn Lemma, Z has a maximal element

Main Proof

Let $\mathcal{F}$ be a family of closed subsets of X with FIP. We will prove that $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$.

Let $\mathcal{H}$ be the maximal family of subsets of X containing containing $\mathcal{F}$ with FIP.

[4] present here the proof of Tychonoff Theorem that is easier to read than the original version in the book

Claim 1 *Let $p_i : (X, \mathcal{T}) \to (X_i, \mathcal{T}_i)$ be the projection mapping. Then,*

$$\bigcap_{H \in \mathcal{H}} \overline{p_i(H)} \neq \varnothing$$

Claim 2 *Since $\bigcap_{H \in \mathcal{H}} \overline{p_i(H)} \neq \varnothing$, let $x_i \in \bigcap_{H \in \mathcal{H}} \overline{p_i(H)}$, we put $x = \prod_{i \in I} x_i$. Then,*

$$x \in \bigcap_{H \in \mathcal{H}} \overline{H}$$

Since $\mathcal{F}$ is a family of closed sets, then $\mathcal{F} \subseteq \mathcal{H}$ implies $\mathcal{F} \subseteq \{\overline{H} : H \in \mathcal{H}\}$. From Claim **??**, $\bigcap_{H \in \mathcal{H}} \overline{H} \neq \varnothing$ implies $\bigcap_{F \in \mathcal{F}} F \supseteq \bigcap_{H \in \mathcal{H}} \overline{H} \neq \varnothing$.

In the reverse direction, notice that the projection mapping is surjective continuous and surjective continuous preserves compactness.

Proof of Claim ??

$\mathcal{H}$ has FIP, then the family $\{p_i(H) : H \in \mathcal{H}\}$ has FIP, so is $\{\overline{p_i(H)} : H \in \mathcal{H}\}$.

$\{\overline{p_i(H)} : H \in \mathcal{H}\}$ is a family of closed subsets of X_i with FIP, since X_i is compact, then $\bigcap_{H \in \mathcal{H}} \overline{p_i(H)} \neq \varnothing$

Proof of Claim ??

Let O be an open set containing x, then O contains a basic open set [5] containing x that is of the form $B = \bigcap_{i \in J} p_i^{-1}(U_i)$ where $x_i \in U_i \in \mathcal{T}_i$ [6] and J is a finite subset of I.

For every $H \in \mathcal{H}$, $x_i \in \overline{p_i(H)}$, so $U_i \cap \overline{p_i(H)} \neq \varnothing$. That implies $p_i^{-1}(U_i) \cap H \neq \varnothing$.

Lemma 17 *Let $S \subseteq X$ that intersects non-trivially every element of $\mathcal{H}$ [7]. Then*

$$S \in \mathcal{H}$$

Lemma 18 *Let $\{H_1, H_2, .., H_n\}$ be a finite subset of $\mathcal{H}$. Then,*

$$H' = \bigcap_{i=1}^{n} H_i \in \mathcal{H}$$

By Lemma **??**, $p_i^{-1}(U_i) \in \mathcal{H}$ for every $i \in I$. By Lemma **??**, $B \in \mathcal{H}$. By FIP on $\mathcal{H}$, for every $H \in \mathcal{H}$, $B \cap H \neq \varnothing$. Hence, $O \cap H \neq \varnothing$

Therefore, x is either contained in or a limit point of every $H \in \mathcal{H}$. Hence $x \in \bigcap_{H \in \mathcal{H}} \overline{H}$

---

[5] the open set in the canonical basis $\mathcal{B}$

[6] $p_j^{-1}(U_j) = U_j \times \prod_{i \in I \setminus X_j} X_i$

[7] $H \in \mathcal{H} \implies H \cap S \neq \varnothing$

Proof of Lemma ??

Suppose $S \notin \mathcal{H}$, $\mathcal{H}$ has FIP implies $\{S\} \cup \mathcal{H}$ has FIP. That violates the maximal assumption of $\mathcal{H}$

Proof of Lemma ??

Suppose $H' \notin \mathcal{H}$, $\mathcal{H}$ has FIP implies $\{H'\} \cup \mathcal{H}$ has FIP. That violates the maximal assumption of $\mathcal{H}$

Next

I STOPPED THIS SERIES HERE UNTIL FURTHER NOTICE DUE TO LACK OF TIME