topology - a categorical approach

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Chapter 1

FOUR CONSTRUCTIONS OF TOPOLOGY

Categorical definitions of some common topologies

Definition 1 (subspace topology)**.** *Let $(X, \mathcal{T}_X)$ be a topological space and $i : A \to X$ be a monomorphism in* Set*. The subspace topology $\mathcal{T}_A$ on A is defined as the coarsest topology such that the map $i : A \to X$ is continuous. Equivalently, the subset topology on A is characterized by universal property as follows: for any morphism $(Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ in* Top*, if there is a lift $Y \to X$ in* Set *then there is also a lift $(Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$ in* Top *such that the diagram below commutes*

$$\begin{tikzcd} & Y \arrow[dl, dashed] \arrow[d] \\ A \arrow[r, hook, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} & (Y, \mathcal{T}_Y) \arrow[dl, dashed] \arrow[d] \\ (A, \mathcal{T}_A) \arrow[r, hook, "i"] & (X, \mathcal{T}_X) \end{tikzcd}$$

Definition 2 (quotient topology)**.** *Let $(X, \mathcal{T}_X)$ be a topological space and $p : X \to B$ be an epimorphism in* Set*. The quotient topology $\mathcal{T}_B$ on B is defined as the finest topology such that the map $p : X \to B$ is continuous. Equivalently, the quotient topology on B is characterized by universal property as follows: for any morphism $(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, if there is a lift $B \to Y$ in* Set *then there is also a lift $(B, \mathcal{T}_B) \to (Y, \mathcal{T}_Y)$ in* Top *such that the diagram below commutes*

$$\begin{tikzcd} X \arrow[r, two heads, "p"] \arrow[d] & B \arrow[dl, dashed] \\ Y \end{tikzcd} \qquad \begin{tikzcd} (X, \mathcal{T}_X) \arrow[r, two heads, "p"] \arrow[d] & (B, \mathcal{T}_B) \arrow[dl, dashed] \\ (Y, \mathcal{T}_Y) \end{tikzcd}$$

Definition 3 (product topology)**.** *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_\alpha$ be an arbitrary collection of topological spaces, $X = \prod_\alpha X_\alpha$, and $p_\alpha : X \to X_\alpha$ be the natural projection. The product topology $\mathcal{T}_X$ on X is defined as the coarsest topology such that every $p_\alpha : X \to X_\alpha$ is continuous. Equivalently, the product topology is characterized by universal property as follows: for any collection of morphisms $\{(Y, \mathcal{T}_Y) \to (X_\alpha, \mathcal{T}_\alpha)\}_\alpha$, if there is a lift $Y \to X$ in* Set *then there is also a lift $(Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ in* Top *such that the diagram below commutes*

$$\begin{tikzcd} & Y \arrow[dl, dashed] \arrow[d] \\ X \arrow[r, two heads, "p_\alpha"] & X_\alpha \end{tikzcd} \qquad \begin{tikzcd} & (Y, \mathcal{T}_Y) \arrow[dl, dashed] \arrow[d] \\ (X, \mathcal{T}) \arrow[r, two heads, "p_\alpha"] & (X_\alpha, \mathcal{T}_\alpha) \end{tikzcd}$$

Definition 4 (coproduct topology)**.** *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_\alpha$ be an arbitrary collection of topological spaces, $X = \coprod_\alpha X_\alpha$, and $i_\alpha : X_\alpha \to X$ be the natural inclusion. The coproduct topology $\mathcal{T}_X$ on X is defined as the finest topology such that every $i_\alpha : X_\alpha \to$ is continuous. Equivalently, the coproduct topology is characterized by universal property as follows: for any collection of morphisms $\{(X_\alpha, \mathcal{T}_\alpha) \to (Y, \mathcal{T}_Y)\}_\alpha$, if there is a lift $X \to Y$ in* Set *then there is also a lift $(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ in* Top *such that the diagram below commutes*

$$\begin{tikzcd} X_\alpha \arrow[r, hook, "i_\alpha"] \arrow[d] & X \arrow[dl, dashed] \\ Y \end{tikzcd} \qquad \begin{tikzcd} (X_\alpha, \mathcal{T}_\alpha) \arrow[r, hook, "i_\alpha"] \arrow[d] & (X, \mathcal{T}_X) \arrow[dl, dashed] \\ (Y, \mathcal{T}_Y) \end{tikzcd}$$

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