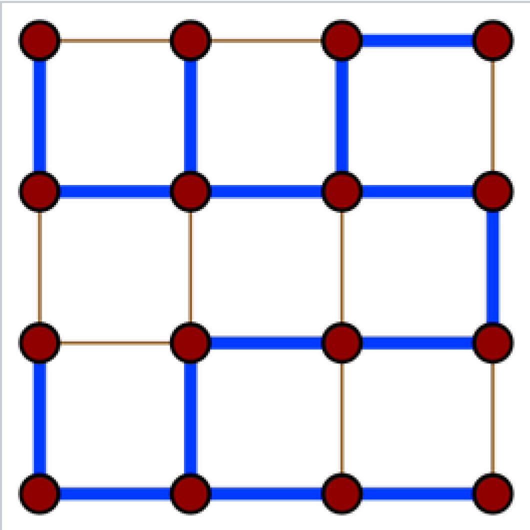
Zorn's Lemma

Nguyen Ngoc Khanh

October 2023



Zorn's lemma can be used to show that every connected graph has a spanning tree. The set of all sub-graphs that are trees is ordered by inclusion, and the union of a chain is an upper bound. Zorn's lemma says that a maximal tree must exist, which is a spanning tree since the graph is connected.[1] Zorn's lemma is not needed for finite graphs, such as the one pictured here.

Definition 1 (Partially Order Set) *A partial order on a set X is a binary relation, denoted by $\leq$ such that*

1. *reflexivity: $x \leq x$ for all $x \in X$*
2. *anti-symmetry: $x = y$ if $x \leq y$ and $y \leq x$ for all $x, y \in X$*
3. *transitivity: $x \leq z$ if $x \leq y$ and $y \leq z$ for all $x, y, z \in X$*

A set X equipped with a partial order $\leq$ is said to be a partially ordered set, denoted by $(X, \leq)$

Definition 2 (Linearly Ordered Set) *A partially ordered set $(X, \leq)$ is said to be a linearly ordered set if every two elements are comparable, that is, $x \leq y$ or $y \leq x$ for all $x, y \in X$. The order $\leq$ is said to be a linear order.*

Definition 3 (Upper Bound) *Given a partially ordered set $(X, \leq)$, an element $t \in X$ is said to be an upper bound of a subset $Y \subseteq X$ if $y \leq t$ for all $y \in Y$*

Definition 4 (Maximal) *Given a partially ordered set $(X, \leq)$, an element $w \in X$ is said to be maximal if $x \in X$ and $w \leq x$, then $x = w$*

Axiom 1 (Zorn's Lemma) *Let $(X, \leq)$ be a non-empty partially ordered set in which every linearly ordered subset has an upper bound. Then, $(X, \leq)$ has a maximal element.*