cycle

nguyenngockhanh.pbc

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Is there an natural number function that always has a cycle and the cycle length is unbounded?

1 Problem

Firstly, let define precisely the notations we are using in this text.

Definition 1 *(Natural Number Set)*

$$\mathbb{N} = \{1, 2, 3, \dots\} \tag{1}$$

Definition 2 *(Natural Number Function) All functions that have the domain and codomain* $\mathbb{N}$

$$f : \mathbb{N} \to \mathbb{N} \tag{2}$$

Denote $f^m(n)$ where $n \in \mathbb{N}$ be the value of m-times recursive call of function f to input n. More formally,

$$\begin{gathered} f^1(n) = f(n), \\ f^{m+1}(n) = f(f^m(n)) \end{gathered}$$

Definition 3 *(Cycle number of a Natural Number Function) Let* $f : \mathbb{N} \to \mathbb{N}$. *A cycle set of* n *is the set of all numbers* $m \in \mathbb{N}$ *such that* $f^m(n) = n$

$$c^{(f)}(n) = \{m : m \in \mathbb{N} \wedge f^m(n) = n\} \tag{3}$$

if cycle set is non-empty, define the cycle number as the smallest number in the cycle set

$$c^{(f)}_{\min}(n) = \min c^{(f)}(n) \tag{4}$$

In this text, we ignore the trivial case where cycle number is 1.

Our main theorem is stated as follows

Theorem 1 *(Cycle) There exists a natural number function such that (1) it has a non-trivial cycle number for every input and (2) the set of all cycle numbers is unbounded.*

$$\exists f : \mathbb{N} \to \mathbb{N}, (\forall n \in \mathbb{N}, c^{(f)}_{\min}(n) > 1) \land (\forall m_0 \in \mathbb{N}, \exists n \in \mathbb{N}, c^{(f)}_{\min}(n) \geq m_0) \tag{5}$$

In other words, this function partitions the natural number set into infinitely number of finite subsets where the cardinality of them are unbounded. Each cycle is associated with a subset.

2 Proof

A simple construction satisfies those properties is as follows:

Suppose we found partition on $\mathbb{N}$ of (1^*) infinitely many finite subsets where (1^{**}) each of them has the cardinality of at least 2, (2^*) the cardinality of these subsets is unbounded.

$$\mathbb{P} = \{P_i\}_{i=1}^{\infty} = \{P_1, P_2, P_3, \dots\} \tag{6}$$

Where we order all elements in each P_i, so that for every P_i, we have a minimum element, a maximum element and a function to yield the successor element if the input is not the maximum element namely

$$succ_i : P_i \backslash \{\max P_i\} \to P_i \backslash \{\min P_i\} \tag{7}$$

Define a function $f_i : P_i \to P_i$ that returns minimum element of P_i if the input is the maximum element of P_i, otherwise return its successor.

$$f_i(n) = \begin{cases} \min P_i & \text{if } n = \max P_i. \\ succ_i(n) & \text{otherwise.} \end{cases} \tag{8}$$

This function has a cycle number of the cardinality of P_i Since $\mathbb{P}$ is a partition, these P_i are disjoint and their union is $\mathbb{N}$. We define the function $f : \mathbb{N} \to \mathbb{N}$

$$f(n) = f_i(n) \text{ if } n \in P_i \tag{9}$$

($1^* \land 1^{**} \to 1$) For every input n, the cycle number is $|P_i| \geq 2$ where P_i is the associated subset. ($2^* \to 2$) Since these subsets are unbounded in size, the set of all cycle numbers of f is also unbounded.

In order to finish the proof, we will show a partition on $\mathbb{N}$ that satisfies (1^*), (1^{**}) and (2^*).

Let S_i be the set of all natural numbers in the range $[2^i, 2^{i+1})$ for $i = 0, 1, 2, \dots$. e.g. $S_0 = \{1\}$, $S_1 = \{2, 3\}$, $S_2 = \{4, 5, 6, 7\}$, $S_3 = \{8, 9, \dots, 15\}$, etc.

Our partition is in the form

$$\mathbb{P} = \{\{S_0 \cup S_1\}\} \cup \{S_i\}_{i=2}^{\infty} \tag{10}$$