Given the problem:

$$\textbf{Minimize: } f(x) \textbf{ subject to: } c_0(x) = \sum_{i=1}^{n} x_i = 1 \text{ and } c_i(x) = -x_i \leq 0 \ \forall i$$

Such that $\frac{\partial f}{\partial x_i}$ has this property:

$$(1)\text{: } \frac{\partial f}{\partial x_i} \text{ is a monotonically increasing function for } x_i \in [0, 1]$$

$$\frac{\partial f}{\partial x_i}(x_1) < \frac{\partial f}{\partial x_i}(x_2) \ \forall\ 0 \leq x_1 < x_2$$

Theorem 1 (Unique solution)**.** *Program has unique solution at* $x_i = \frac{1}{n}\ \forall i$

Lagrangian function is

$$L(x, \mu, \lambda) = f(x) + \sum_{i=1}^{n} \mu_i c_i(x) + \lambda c_0(x)$$

If $x*$ is a minimum, KKT conditions:

Stationary: $\frac{\partial L}{\partial x}(x*) = 0_n$

Primal feasibility: $c_0(x*) = 0$ and $c_i(x*) \leq 0\ \forall i$

Dual feasibility: $\mu_i \geq 0\ \forall i$

Complementary slackness: $\mu_i c_i(x*) = 0\ \forall i$

We have:

$$\frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} - \mu_k + \lambda$$

Consider 2 cases:

Case 1: all $x_i > 0$, due to complementary slackness, all $\mu_i = 0$. So that, all $\frac{\partial f}{\partial x_i}$ must be equal. (1) deduces that the unique solution satisfying KKT conditions is $x_i = \frac{1}{n}$

Case 2: some $x_i = 0$, let $x_{i1} = 0$

$$\frac{\partial f}{\partial x_i}(x_{i1}) - \mu_{i1} + \lambda = 0$$

$$\lambda = \mu_{i1} - \frac{\partial f}{\partial x_i}(0)$$

Since dual feasibility, $\mu_{i1} \geq 0$, So

$$\lambda \geq -\frac{\partial f}{\partial x_i}(0)$$

There is at least one $x_i > 0$, let $x_{i2} > 0$, due to complementary slackness, $\mu_{i2} = 0$, So

$$\frac{\partial f}{\partial x_i}(x_{i2}) + \lambda = 0$$

$$\frac{\partial f}{\partial x_i}(x_{i2}) = -\lambda \leq \frac{\partial f}{\partial x_i}(0)$$

(1) deduces contradiction.