Two-Sum

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1 Preliminaries

Let first define some data structures

Denote $[a, b] = \{a, a+1, a+2, ..., b-1, b\}$ for $a < b \in \mathbb{N}$ be the set of all natural numbers between a and b (inclusive)

Definition 1 (finite length array) *An array of length n of element from S is defined as a function $f : [1, n] \to S$*

The set all all arrays of length n from S is $S^{[1,n]}$

The set of all arrays from S is $\bigcap_{n=0}^{\infty} S^{\mathbb{N} \cap [1,n]}$ or S^* for short.

Definition 2 (sub-array) *Let $a \leq b \in \mathbb{N} \cap [1, n]$, the sub-array of f with respect to bounds (a, b) denoted as $f_{(a,b)} : \mathbb{N} \cap [1, b-a+1] \to S$ is defined by $f_{(a,b)}(i) = f(a+i-1) \forall i \in \mathbb{N} \cap [1, b-a+1]$*

Note that $f_{(1,n)} = f$

Definition 3 (increasing array) *An array $f : \mathbb{N} \cap [1, n] \to \mathbb{N}$ is increasing if and only if $\forall i < j \in \mathbb{N} \cap [1, n], f(i) \leq f(j)$*

Definition 4 (array membership) *If $\exists i \in \mathbb{N}, x = f(i)$, we write $x \in f$*

Definition 5 (array length) *The function $len : S^* \to \mathbb{N}$ returns the length of an array*

2 Two-Sum

Given an increasing array f of length n and a number s, if there exists two numbers $a, b \in f$ such as $a + b = s$, define a function $ts : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N} \times \mathbb{N}$ as $ts(s, f) = (a, b)$ where $a \leq b \in f$ and $a + b = s$

One possible definition of ts is as follow:

$$ts(s, f) = \begin{cases} (f(1), f(n)) & \text{if } f(1) + f(n) = s \\ ts(s, f_{(2,n)}) & \text{if } f(1) + f(n) < s \\ ts(s, f_{(1,n-1)}) & \text{if } s < f(1) + f(n) \end{cases} \tag{1}$$